A Calculus for Ressource Control of Components

P. Brambilla

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Abstract

This paper describes a calculus for the resource control of Petri net components. In order to calculate the new input/output behavior of a component resulting from a composition we have developed a formalism for a very restricted set of components. We first introduce the class of circular components and define their composition in our formalism. Afterwards we give a calculus with simplification rules for these components and show that each component has a unique normal form.

CR Categories and Subject Descriptors: I.2.3 [Deduction and Theorem Proving]: Deduction (rule-based)

General Terms: Petri nets, Normalization

Additional Key Words: Components

1 Introduction

There exist a lot of tools where different kind of processes can be modeled and simulated. Petri nets have become a quite in this scope. One of these tools called BusinessSpecs was developed by the IvyTeam Zug and is mainly used to model business processes. Although the basic elements of Petri nets are very simple, it is possible to build very complex nets. One of the aims of BusinessSpecs was to structure more complex nets by grouping subnets into Petri net components. These components communicate through only a few inputs and outputs with their environment. It was now necessary to have a calculus where the behavior of these components could be modeled.

In this paper we present a calculus for the resource control of a restricted set of such components.

We first give some auxiliary definitions and then define cyclic components and their composition. At the end we present a calculus used to simplify and normalize cyclic components.

2 Auxiliary Definitions

In this section we give some auxiliary definitions that are used in section 3 for the definition of components. These definitions mainly concern sequences of pairs of natural numbers, which are used for the description of the I/O-behavior of components.

2.1 Definition ($\mathcal{P}_{\mathcal{M}}(A)$)
For a set A we define $\mathcal{P}_{\mathcal{M}}(A)$ as the set of multisets on A.

2.2 Definition ($\mathbb{S}$, $\mathbb{S}_{\langle\rangle}$)
We write $\mathbb{S}_{\langle\rangle}$ and $\mathbb{S}$ to denote the set of finite (not empty) sequences of pairs of natural numbers.

$$\begin{aligned} \mathbb{S}_{\langle\rangle} &:= \{\langle(x_1,y_1)\dots(x_n,y_n)\rangle \mid n \in \mathbb{N}_0 \text{ and } x_i, y_i \in \mathbb{N}^+ \text{ for } 1 \le i \le n\} \\ \mathbb{S} &:= \mathbb{S}_{\langle\rangle} \setminus \{\langle\rangle\} \end{aligned}$$

We use the symbols σ, τ and ρ with or without indices to denote elements of $\mathbb{S}_{\langle\rangle}$.

2.3 Definition (Concatenation)
For two sequences $\sigma_1 := \langle(x_1,y_1)\dots(x_n,y_n)\rangle$ and $\sigma_2 := \langle(u_1,v_1)\dots(u_m,v_m)\rangle$ we define the concatenation $\sigma_1 \circ \sigma_2$ of σ_1 and σ_2 as

$$\sigma_1 \circ \sigma_2 := \langle(x_1,y_1)\dots(x_n,y_n)(u_1,v_1)\dots(u_m,v_m)\rangle$$

For $S \subset \mathbb{S}$ and $T \subset \mathbb{S}$ we define

$$S \circ S := \{\sigma \circ \tau \mid \sigma \in S, \tau \in S\}$$

Remark: For $S_1, S_2, S \subset \mathbb{S}$ the following holds: $(S_1 \cup S_2) \circ S = S_1 \circ S \cup S_2 \circ S$.

2.4 Definition ($\mathrm{len}(\sigma)$)
For a sequence $\sigma \in \mathbb{S}_{\langle\rangle}$ we define its *length* $\mathrm{len}(\sigma)$ inductively:

$$\begin{aligned} \mathrm{len}(\langle\rangle) &= 0 \\ \mathrm{len}(\sigma \circ \langle(x,y)\rangle) &= \mathrm{len}(\sigma) + 1 \end{aligned}$$

2.5 Definition ($\pi(\sigma)$)
For $\sigma = \langle (x_1, y_1) \dots (x_n, y_n) \rangle \in \mathbb{S}$ we define

$$\pi(\sigma) := \prod_{i=1}^{n} \frac{x_i}{y_i}$$

3 Components

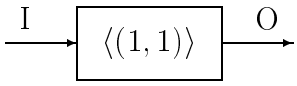
In this section we give the definition of components that reach their initial state after a certain amount of data tokens has entered.
Afterwards we define the composition of such components, present simplification rules for their I/O-behavior and show that these rules have the Church-Rosser property.

3.1 Cyclic component

From the user's view, a component consists of communication gates where data tokens enter or leave the component, and a description of the behavior of the component (i.e. how it reacts on a given input).

3.1.1 A simple component

Regarding the communication gates we distinguish between input gates and output gates and often just call them inputs and outputs. The most simple component consists of exactly one input and one output and emits for each token that enters the component a token at its output. This I/O-behavior is denoted with the sequence $\langle (1, 1) \rangle$.



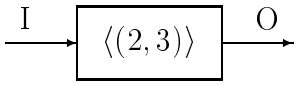
3.1.2 Adding weights

Of course it is possible that more than one token must enter at an input gate for a reaction to take place at the output of a component. As an example we modify the component above so that 2 tokens must enter at I for 1 token to be emitted at the output O.

There are two possibilities to add weights to the I/O-behavior of the component. You can either attribute the input of the component with the weight or you can put the weight into the sequence of the I/O-behavior.
Since we already use sequences to describe the I/O-behavior it seems best to put the weights into this sequences. We will see later that attributing the inputs with weights is useful for synchronized inputs.

It is also reasonable that a component reacts to a given input by outputting more than one token at an output and therefore the sequence can be weighted the same way for the output.

A component that needs 2 tokens at its input to emit 3 tokens at its output would therefore be denoted with the sequence $\langle (2, 3) \rangle$.



Components with exactly one input and one output are called *stream components*. A stream component is *atomic* if the length of its only sequence is 1.

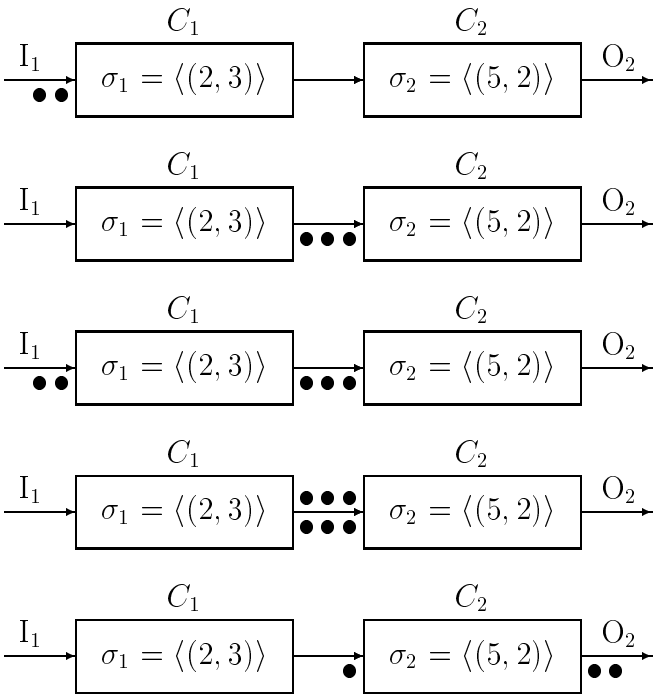
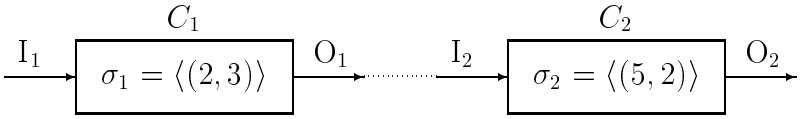


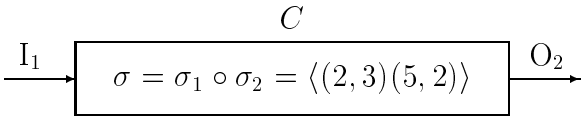
Figure 1: A simple composition

3.1.3 A simple composition

To see the basic idea of the composition we have a look at the composition of two atomic stream components:



From this connection a new component C with I_1 as input and O_2 as output results. The sequence σ describing the behavior is simply the concatenation of σ_1 and σ_2.

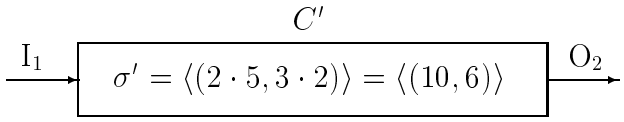


On this example we can show what it means if the I/O-behavior is denoted with a sequence consisting of more than one pair of natural numbers.

Suppose that 2 tokens enter C at I_1. C_1 (the first part of C) emits 3 tokens but these are not enough tokens for C_2 to react. It takes 2 more tokens at I_1 for a reaction to take place at C_2 resulting in 2 tokens emitted at O_2. However, since there is still one token in C_2 that has not yet been processed – only 5 of the 6 tokens emitted by C_1 were used – component C is not yet in its initial state (see figure 1). It takes 10 tokens at I_1 for the component to complete one cycle and reach its initial state.

One is tempted to use the I/O-ratio to describe the behavior of the component above. The I/O-ratio of C_1 is 2/3 (2 tokens are needed for 3 tokens to be emitted), the one of

C_2 is 5/2. Multiplying these ratios gives the I/O-ratio of the composition (10/6). Using the numerator and denominator of this fraction to form σ leads to the component C':

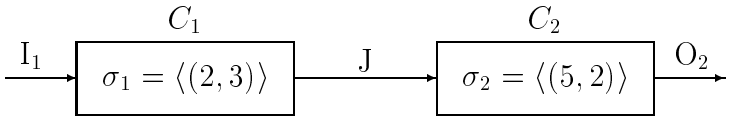


The I/O-ratio of C' is correct but σ' does not reflect the correct behavior. According to this specification it would need 10 tokens entering the component for 6 tokens to be emitted. An input of less than 10 tokens would cause no reaction, which is obviously wrong. Canceling down the ratio 10/6 to 5/3 does not result in a sequence describing the correct behavior either.

There is no way to describe the I/O-behavior of C with a sequence of length 1. This is due to the fact that the additional pair in the sequence represents an implicit internal state.

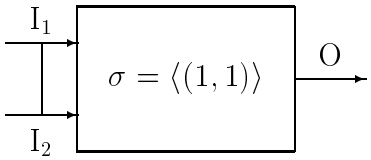
The state of an atomic stream component can be described by a natural number that reflects the number of tokens that have entered the component but that have not yet been processed. This kind of description does not work anymore with the composition C because the unprocessed tokens of its second part can not be expressed in that manner. However, both tuples in σ represent an atomic stream component and therefore assigning a natural number to each tuple is sufficent to describe the state of the component.

Generally each additional pair in a sequence represents an implicit internal state J:

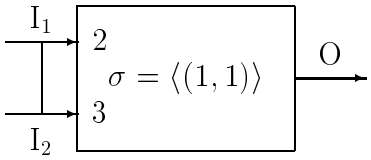


3.1.4 Synchronized inputs

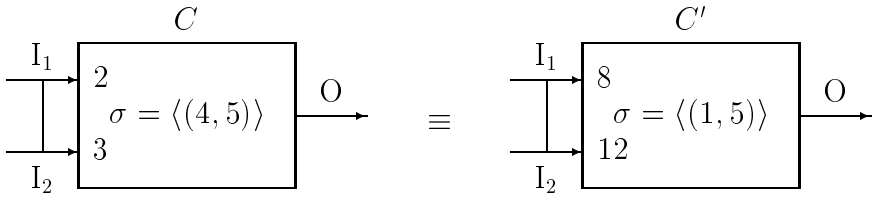
One of the essential concepts for modeling processes is the possibility to synchronize inputs. Synchronization of input gates means that a token must be present on each gate for the component to react. In the graphical representation of components we denote this by drawing a line between the input gates:



Synchronized inputs may also be weighted. Putting the weights into the sequence has the disadvantage that the inputs can not be weighted differently. For example you can not model a component that outputs a token at O if 2 and 3 tokens have entered at I_1 and I_2, respectively. As mentioned above it is therefore necessary to attribute inputs with weights:



Both weighting methods can be combined, i.e. you can attribute the inputs with weights and put weights into the sequence. This can lead to components that look different but behave the same. For example have a look at the following two components:

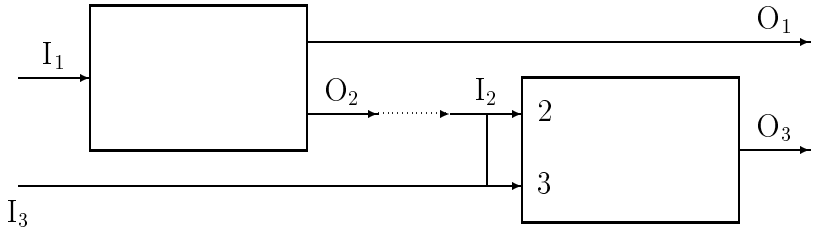


In C 4 times the weighted input (2 tokens at I_1 and 3 tokens at I_2) is necessary for 5 tokens to be emitted. The behavior is equal to the one of C' where the weighted input (8 tokens at I_1 and 12 tokens at I_2) is required only once to emit 5 tokens.

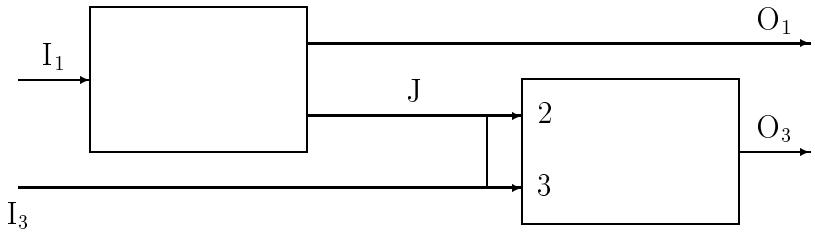
To prevent that kind of ambiguities we demand that the greatest common divider of the weights of synchronized inputs is 1.

3.1.5 Internal states

Besides the implicit internal state represented by pairs in sequences we need to have explicit internal states if an output of a component is connected to a synchronized input, as in the following example.

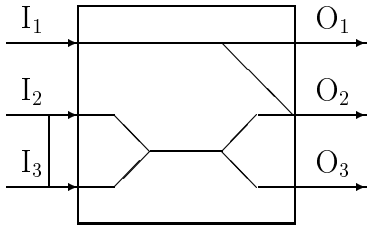


Data tokens entering at I_1 result in tokens at O_1 and O_2. The tokens emitted at O_2 are fed to the synchronized input I_2 of the second part of the component. These tokens can not be processed until there are enough tokens given to I_3. Therefore we keep book about the number of tokens remaining at I_2 by introducing an internal state J for I_2:



3.1.6 Assignment of Inputs to Outputs

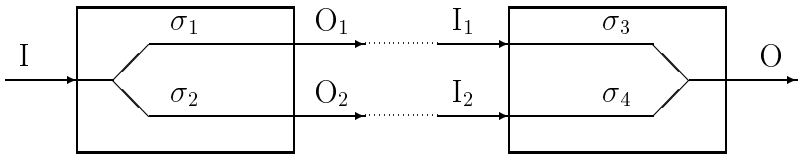
In all examples components above where we specified a sequence for the description of the I/0-behavior we only had one input or two synchronized inputs and one output. It was therefore clear that the given sequence described the I/O-behavior between all inputs and the output. Generally there are several inputs and outputs where each input is connected to some – but not necessarily all – outputs of a component. Consider the following component where tokens entering at I_1 result in tokens at O_1 and O_2, and tokens entering the synchronized inputs I_2 and I_3 result in tokens at O_2 and O_3.



The outputs O_1 and O_2 need not have the same I/O-behavior. It is for example possible that O_1 emits a token for every token entering at I_1 and O_2 emits a token for every other token entering at I_1. Therefore a sequence concerning the input I_1 is assigned to each of them.
Furthermore O_2 also reacts on tokens entering at the synchronized inputs I_2 and I_3. A sequence concerning I_2 and I_3 is therefore assigned to O_2, too.
To be able to model synchronized inputs the assignment of sequences to outputs is done in dependence of the set holding the synchronized inputs. A possible assignment is given in the following table:

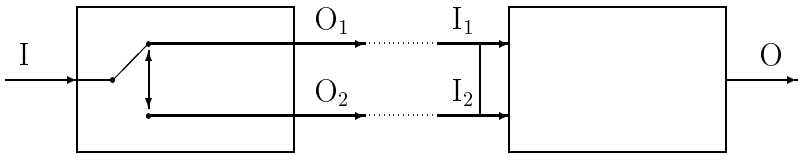
	O_1	O_2	O_3
$\{I_1\}$	σ_{11}	σ_{12}	—
$\{I_2, I_3\}$	—	σ_{22}	σ_{23}

The table above represents only a possible assignment because it is in fact possible, that two or more sequences concerning the same set of inputs are assigned to an output. Consider the following composition:



I is connected in two independent ways to O – via I_1 and via I_2. The connection via I_1 results in the sequence $\sigma_1 \circ \sigma_3$, the other connection results in $\sigma_2 \circ \sigma_3$. Therefore the multiset of sequences $S = \{\sigma_1 \circ \sigma_3, \sigma_2 \circ \sigma_3\}$ is assigned to O in conjunction with the set $\{I\}$. We use multisets of sequences instead of sets because the multiplicity of elements is important to us. If in the above example σ_1 is equal to σ_2 and σ_3 is equal to σ_4 and we would use sets instead of multisets then $\{\sigma_1 \circ \sigma_3, \sigma_2 \circ \sigma_4\}$ would consists of only one element and the component would produce only half the tokens that it should.

For a component to be cyclic it is important that the reaction on a given input is unique. It is for example not possible that a token entering at an input I may result in either a token at an output O_1 or at an output O_2. To see why this is forbidden consider the following example:



Since it is not determined which of the outputs O_1 and O_2 reacts on an input at I it is possible that O_2 never emits a token, with the effect that O will never emit a token and the component never reaches its initial state again. It is also possible that the tokens are

emitted at O_1 and O_2 in a way that there is always at least one token at I_1 or I_2 that has not yet been processed.

We are now able to give a formal definition of a cyclic component.

3.1 Definition (Cyclic component)
A *cyclic component* is defined as a tuple $C := (\mathcal{I}, \mathcal{O}, \mathcal{J}; \mathcal{V}, \mathbf{f})$ with

$$\begin{array}{rcl} \mathcal{I} & := & \{\mathrm{I}_1, \ldots, \mathrm{I}_{n_\mathrm{I}}\} \\ \mathcal{O} & := & \{\mathrm{O}_1, \ldots, \mathrm{O}_{n_\mathrm{O}}\} \\ \mathcal{J} & := & \{\mathrm{J}_1, \ldots, \mathrm{J}_{n_\mathrm{J}}\} \\ \mathcal{V} & := & \{(\mathrm{V}_i, \mathrm{W}_i, f_i) \mid 1 \leq i \leq n_\mathrm{V}\} \\ \mathbf{f} & : & \mathcal{I} \cup \mathcal{J} \to \mathbb{N}^+ \end{array}$$

$\mathcal{I}$, $\mathcal{O}$ and $\mathcal{J}$ represent the sets of inputs, outputs and internal states, respectively. $\mathbf{f}$ is the function that assigns a weight to each input or internal state, $\mathcal{V}$ specifies the mapping of multisets of sequences to outputs in regard of sets of inputs. For $\mathbf{f}$ and $\mathcal{V}$ the following must hold:

$$\forall(\mathrm{V}, \mathrm{W}, f) \in \mathcal{V}.(\gcd(\{\mathbf{f}(\mathrm{I}) \mid \mathrm{I} \in \mathrm{V}\}) = 1)$$

$$\bigcup \mathrm{V}_i = \mathcal{I} \cup \mathcal{J} \qquad i \neq j \Longrightarrow \mathrm{V}_i \cap \mathrm{V}_j = \emptyset$$

$$\bigcup \mathrm{W}_i = \mathcal{O} \cup \mathcal{J}$$

$$f_i : \mathcal{O} \cup \mathcal{J} \to \mathcal{P}_\mathcal{M}(\mathbb{S}) \qquad f_i(X) = \emptyset \text{ if } X \notin \mathrm{W}_i$$

The I/O-behavior of a component is defined by the following terms:

$$\sum_{A \in \mathrm{V}_h} \mathbf{f}(A)A \to \sum_{B \in \mathrm{W}_h} \sum_{\sigma \in f_h(B)} \sigma B \qquad (1 \leq h \leq n_\mathrm{V}).$$

That kind of terms are called *B-terms*.

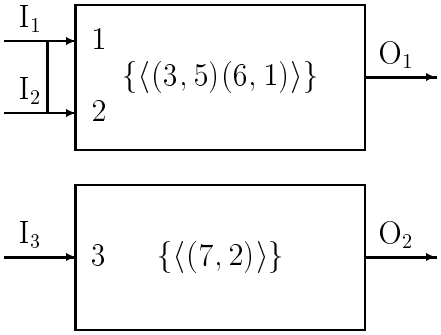
We now have a look at an example component with two synchronized inputs and one output:

3.2 Example

$$C = (\{\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3\}, \{\mathrm{O}_1, \mathrm{O}_2\}, \emptyset; \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{O}_1\}, f_1), (\{\mathrm{I}_3\}, \{\mathrm{O}_2\}, f_2), \}, \mathbf{f})$$

where

$$f_1(\mathrm{O}_1) = \{\langle(3,5)(6,1)\rangle\}, \quad f_2(\mathrm{O}_2) = \{\langle(7,2)\rangle\}, \quad f_1(\mathrm{O}_2) = f_2(\mathrm{O}_1) = \emptyset$$
$$\mathbf{f}(\mathrm{I}_1) = 1, \quad \mathbf{f}(\mathrm{I}_2) = 2, \quad \mathbf{f}(\mathrm{I}_3) = 3$$



This example also makes it clear that a component can consists on several independent parts.

The B-terms of this component are:

$$\begin{aligned} 1\mathrm{I}_1 + 2\mathrm{I}_2 &\rightarrow \langle (3,5),(6,1)\rangle \mathrm{O}_1 \\ 3\mathrm{I}_3 &\rightarrow \langle (7,2)\rangle \mathrm{O}_2 \end{aligned}$$

3.3 Definition (Ground component)
A cyclic component $(\mathcal{I}, \mathcal{O}, \mathcal{J}; \mathcal{V}, \mathbf{f})$ is called *ground component*, if $\mathcal{J} = \emptyset$.

3.4 Definition (Independent and dependent input)
Let $C := (\mathcal{I}, \mathcal{O}, \mathcal{J}; \mathcal{V}, \mathbf{f})$ be a cyclic component.
An input or internal state $\mathrm{I} \in \mathcal{I} \cup \mathcal{J}$ is called *independent*, if the following holds:

$$(\exists (\mathrm{V}, \mathrm{W}, f) \in \mathcal{V})(\mathrm{V} = \{\mathrm{I}\}),$$

otherwise it is called *dependent*.

An output or internal state $X \in \mathcal{O} \cup \mathcal{J}$ is said to *depend on an input* $\mathrm{I} \in \mathcal{I}$ ($\mathrm{dep}(X, \mathrm{I})$) if the following holds:

$$(\exists (\mathrm{V}, \mathrm{W}, f) \in \mathcal{V})\big(X \in \mathrm{W} \wedge (\mathrm{I} \in \mathrm{V} \vee (\exists Y \in \mathrm{V})\, \mathrm{dep}(Y, \mathrm{I}))\big)$$

3.2 Composition

We now give the formal definition of the composition of an output to an input. Note that we do not allow loops in the composition. Therefore the connection from an output to an input it depends on is forbidden.

3.5 Definition (Composition)
Let $C := (\mathcal{I}, \mathcal{O}, \mathcal{J}; \mathcal{V}, \mathbf{f})$ be a cyclic component.

The connection of an output $\mathrm{O} \in \mathcal{O}$ to an Input $\mathrm{I} \in \mathcal{I}$ results in a new cyclic component

$$C' := (\mathcal{I}', \mathcal{O}', \mathcal{J}'; \mathcal{V}', \mathbf{f}')$$

Assume $\mathrm{I} \in \mathrm{V}_k$.

1. O independent of I.
   (a) I independent input (i.e. $\mathrm{V}_k = \{\mathrm{I}\}$).

$$\begin{aligned} \mathcal{I}' &:= \mathcal{I} \setminus \{\mathrm{I}\} \\ \mathcal{O}' &:= \mathcal{O} \setminus \{\mathrm{O}\} \\ \mathcal{J}' &:= \mathcal{J} \\ \mathcal{V}' &:= \{(\mathrm{V}_i, \mathrm{W}'_i, f'_i) \mid 1 \le i \le n_{\mathrm{V}}, i \neq k\} \text{ with} \\ \mathrm{W}'_i &:= \begin{cases} (\mathrm{W}_i \setminus \{\mathrm{O}\}) \cup \mathrm{W}_k & : \mathrm{O} \in \mathrm{W}_i \\ \mathrm{W}_i & : \mathrm{O} \notin \mathrm{W}_i \end{cases} \\ f'_i(X) &:= f_i(X) \cup f_i(\mathrm{O}) \circ \{\langle (f(\mathrm{I}), 1)\rangle\} \circ f_k(X) \\ \mathbf{f}' &:= \mathbf{f}\restriction_{\mathcal{I}' \cup \mathcal{J}'} \end{aligned}$$

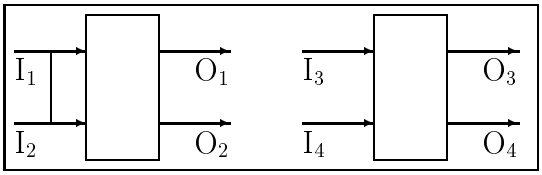
(b) I dependent input.

$$
\begin{array}{rcl}
\mathcal{I}' & := & \mathcal{I} \setminus \{\mathrm{I}\} \\
\mathcal{O}' & := & \mathcal{O} \setminus \{\mathrm{O}\} \\
\mathcal{J}' & := & \mathcal{J} \cup \{\mathrm{I}\} \\
\mathcal{V}' & := & \{(\mathrm{V}_i, \mathrm{W}'_i, f'_i) \mid 1 \leq i \leq n_{\mathrm{V}}\} \text{ with} \\
& & \mathrm{W}'_i \;:=\; \begin{cases} (\mathrm{W}_i \setminus \{\mathrm{O}\}) \cup \{\mathrm{I}\} : \mathrm{O} \in \mathrm{W}_i \\ \mathrm{W}_i \qquad\qquad\qquad\quad : \mathrm{O} \notin \mathrm{W}_i \end{cases} \\
& & f'_i(X) \;:=\; \begin{cases} f_i(\mathrm{O}) : X = \mathrm{I} \\ f_i(X) : X \neq \mathrm{I} \end{cases} \\
\mathbf{f}' & := & \mathbf{f}
\end{array}
$$

2. O depends on I.
   That kind of composition is forbidden.

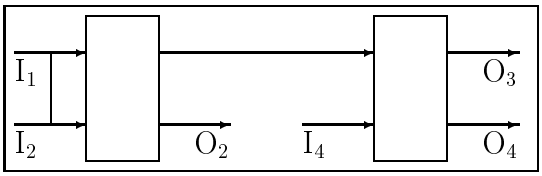
To have a closer look at the composition above we now give two examples where two connections to non synchronized and synchronized inputs are established. The compositions are done step by step in order to see their intermediate states.

Example 1:



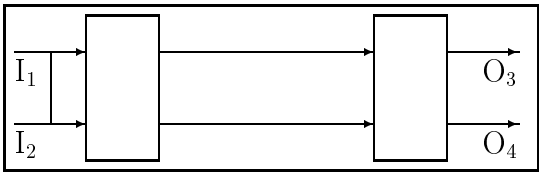
$$
\begin{array}{rcl}
\mathcal{I} & = & \{\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4\} \\
\mathcal{O} & = & \{\mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3, \mathrm{O}_4\} \\
\mathcal{J} & = & \{\} \\
\mathcal{V} & = & \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{O}_1, \mathrm{O}_2\}, f_1), \\
& & \;(\{\mathrm{I}_3\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_2), \\
& & \;(\{\mathrm{I}_4\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_3)\} \\
f_1 & = & \mathrm{O}_1 \mapsto \{\langle(6,7)\rangle\}, \\
& & \mathrm{O}_2 \mapsto \{\langle(8,9)\rangle\} \\
f_2 & = & \mathrm{O}_3 \mapsto \{\langle(10,11)\rangle\}, \\
& & \mathrm{O}_4 \mapsto \{\langle(12,13)\rangle\} \\
f_3 & = & \mathrm{O}_3 \mapsto \{\langle(14,15)\rangle\}, \\
& & \mathrm{O}_4 \mapsto \{\langle(16,17)\rangle\} \\
\mathbf{f} & : & \mathrm{I}_1 \mapsto 2, \quad \mathrm{I}_2 \mapsto 3, \quad \mathrm{I}_3 \mapsto 4, \quad \mathrm{I}_4 \mapsto 5
\end{array}
$$

Connect O_1 to I_3



$$
\begin{array}{lcl}
\mathcal{I}' & = & \{\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_4\} \\
\mathcal{O}' & = & \{\mathrm{O}_2, \mathrm{O}_3, \mathrm{O}_4\} \\
\mathcal{J}' & = & \{\} \\
\mathcal{V}' & = & \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{O}_2, \mathrm{O}_3, \mathrm{O}_4\}, f_1'), \\
 & & \ (\{\mathrm{I}_4\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_3')\} \\
f_1' & = & \mathrm{O}_2 \mapsto \{\langle(8,9)\rangle\}, \\
 & & \mathrm{O}_3 \mapsto \{\langle(6,7)(2,1)(10,11)\rangle\} \\
 & & \mathrm{O}_4 \mapsto \{\langle(6,7)(2,1)(12,13)\rangle\} \\
f_3' & = & \mathrm{O}_3 \mapsto \{\langle(14,15)\rangle\}, \\
 & & \mathrm{O}_4 \mapsto \{\langle(16,17)\rangle\} \\
\mathbf{f}' & : & \mathrm{I}_1 \mapsto 2, \quad \mathrm{I}_2 \mapsto 3, \quad \mathrm{I}_4 \mapsto 5
\end{array}
$$

Connect O_2 to I_4

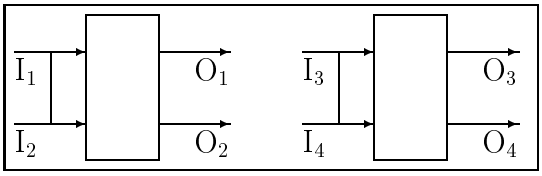


$$
\begin{array}{lcl}
\mathcal{I}'' & = & \{\mathrm{I}_1, \mathrm{I}_2\} \\
\mathcal{O}'' & = & \{\mathrm{O}_3, \mathrm{O}_4\} \\
\mathcal{J}'' & = & \{\} \\
\mathcal{V}'' & = & \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_1'')\} \\
f_1'' & = & \mathrm{O}_3 \mapsto \{\langle(6,7)(2,1)(10,11)\rangle, \langle(8,9)(5,1)(14,15)\rangle\} \\
 & & \mathrm{O}_4 \mapsto \{\langle(6,7)(2,1)(12,13)\rangle, \langle(8,9)(5,1)(16,17)\rangle\} \\
\mathbf{f}'' & : & \mathrm{I}_1 \mapsto 2, \quad \mathrm{I}_2 \mapsto 3
\end{array}
$$

The behavior of this component is expressed by the following B-term:

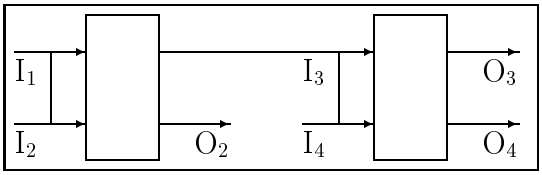
$$
\begin{array}{lcl}
2\mathrm{I}_1 + 3\mathrm{I}_2 & \rightarrow & \langle(6,7)(2,1)(10,11)\rangle \mathrm{O}_3 + \langle(8,9)(5,1)(14,15)\rangle \mathrm{O}_3 + \\
 & & \langle(6,7)(2,1)(12,13)\rangle \mathrm{O}_4 + \langle(8,9)(5,1)(16,17)\rangle \mathrm{O}_4
\end{array}
$$

Example 2:



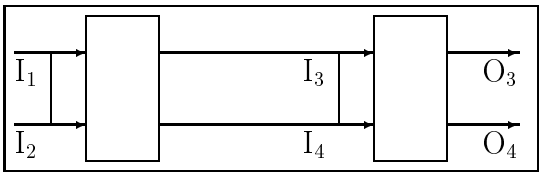
$$
\begin{aligned}
\mathcal{I} &= \{\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4\} \\
\mathcal{O} &= \{\mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3, \mathrm{O}_4\} \\
\mathcal{J} &= \{\} \\
\mathcal{V} &= \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{O}_1, \mathrm{O}_2\}, f_1), \\
&\quad\ (\{\mathrm{I}_3, \mathrm{I}_4\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_2)\} \\
f_1 &= \mathrm{O}_1 \mapsto \{\langle(1,3)\rangle\}, \\
&\quad\ \mathrm{O}_2 \mapsto \{\langle(2,3)\rangle\} \\
f_2 &= \mathrm{O}_3 \mapsto \{\langle(x_3,y_3)\rangle\}, \\
&\quad\ \mathrm{O}_4 \mapsto \{\langle(x_4,y_4)\rangle\} \\
\mathbf{f} &: \quad \mathrm{I}_1 \mapsto 2, \quad \mathrm{I}_2 \mapsto 3, \quad \mathrm{I}_3 \mapsto 3, \quad \mathrm{I}_4 \mapsto 2
\end{aligned}
$$

Connect O_1 to I_3



$$
\begin{aligned}
\mathcal{I}' &= \{\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_4\} \\
\mathcal{O}' &= \{\mathrm{O}_2, \mathrm{O}_3, \mathrm{O}_4\} \\
\mathcal{J} &= \{\mathrm{I}_3\} \\
\mathcal{V}' &= \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{O}_2, \mathrm{I}_3\}, f_1'), \\
&\quad\ (\{\mathrm{I}_3, \mathrm{I}_4\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_2')\} \\
f_1' &= \mathrm{O}_2 \mapsto \{\langle(2,3)\rangle\}, \\
&\quad\ \mathrm{I}_3 \mapsto \{\langle(1,3)\rangle\} \\
f_2' &= \mathrm{O}_3 \mapsto \{\langle(x_3,y_3)\rangle\}, \\
&\quad\ \mathrm{O}_4 \mapsto \{\langle(x_4,y_4)\rangle\} \\
\mathbf{f}' &= \mathrm{I}_1 \mapsto 2, \quad \mathrm{I}_2 \mapsto 3, \quad \mathrm{I}_3 \mapsto 3, \quad \mathrm{I}_4 \mapsto 2
\end{aligned}
$$

Connect O_2 to I_4



$$
\begin{array}{lcl}
\mathcal{I}'' & = & \{\mathrm{I}_1, \mathrm{I}_2\} \\
\mathcal{O}'' & = & \{\mathrm{O}_3, \mathrm{O}_4\} \\
\mathcal{J}'' & = & \{\mathrm{I}_3, \mathrm{I}_4\} \\
\mathcal{V}'' & = & \{(\{\mathrm{I}_1, \mathrm{I}_2\}, \{\mathrm{I}_3, \mathrm{I}_4\}, f_1'') \\
 & & \ (\{\mathrm{I}_3, \mathrm{I}_4\}, \{\mathrm{O}_3, \mathrm{O}_4\}, f_2'')\} \\
f_1'' & = & \mathrm{I}_3 \mapsto \{\langle(1,3)\rangle\}, \\
 & & \mathrm{I}_4 \mapsto \{\langle(2,3)\rangle\} \\
f_2'' & = & \mathrm{O}_3 \mapsto \{\langle(x_3, y_3)\rangle\}, \\
 & & \mathrm{O}_4 \mapsto \{\langle(x_4, y_4)\rangle\} \\
\mathbf{f}'' & = & \mathrm{I}_1 \mapsto 2, \quad \mathrm{I}_2 \mapsto 3, \quad \mathrm{I}_3 \mapsto 3, \quad \mathrm{I}_4 \mapsto 2
\end{array}
$$

The behavior of this component is expressed by the following two B-terms:

$$
\begin{array}{lcl}
2\mathrm{I}_1 + 3\mathrm{I}_2 & \to & \langle(1,3)\rangle \mathrm{I}_3 + \langle(2,3)\rangle \mathrm{I}_4 \\
2\mathrm{I}_3 + 3\mathrm{I}_4 & \to & \langle(x_3, y_3)\rangle \mathrm{O}_3 + \langle(x_4, y_4)\rangle \mathrm{O}_4
\end{array}
$$

For the composition it is important that the resulting component does not depend on the order in which the connections are established. As shown below this holds for our definition of components and composition:

3.6 Theorem
The composition order is not relevant for the resulting component.

⌜

Let $C := (\mathcal{I}, \mathcal{O}, \mathcal{J}; \mathcal{V}, \mathbf{f})$ be a cyclic component. It suffices to show that two compositions can be permuted. Let $(\mathrm{V}_k, \mathrm{W}_k, f_k) \in \mathcal{V}, (\mathrm{V}_l, \mathrm{W}_l, f_l) \in \mathcal{V}$. Without loss of generality let $\mathrm{I}_1 \in \mathrm{V}_k, \mathrm{I}_2 \in \mathrm{V}_l, \mathrm{O}_1, \mathrm{O}_2 \in \mathcal{O}$. We connect O_1 to I_1 and O_2 to I_2 and show that the resulting component is independent of which composition is done first. We define $i_1 := \mathbf{f}(\mathrm{I}_1)$ and $i_2 := \mathbf{f}(\mathrm{I}_2)$. Because of the composition conditions the following holds.

$$\mathrm{O}_1 \notin \mathrm{W}_k \tag{1}$$

$$\mathrm{O}_2 \notin \mathrm{W}_l \tag{2}$$

$$\mathrm{O}_1 \notin \mathrm{W}_l \vee \mathrm{O}_2 \notin \mathrm{W}_k \tag{3}$$

Case 1 I_1 and I_2 independent. Then $k \neq l, \mathrm{V}_k = \{\mathrm{I}_1\}, \mathrm{V}_l = \{\mathrm{I}_2\}$ holds.
- First $\mathrm{O}_1 \to \mathrm{I}_1$, then $\mathrm{O}_2 \to \mathrm{I}_2$

$$
\begin{array}{rclcrclcrcl}
\mathcal{I}' & = & \mathcal{I} \setminus \{\mathrm{I}_1\} & & \mathcal{O}' & = & \mathcal{O} \setminus \{\mathrm{O}_1\} & & \mathcal{J}' & = & \mathcal{J} \\
\mathcal{V}' & = & \{(\mathrm{V}_i, \mathrm{W}'_i, f'_i) \mid 2 \leq i \leq n_\mathrm{V}\} \text{ where} & & & & & & & & \\
\mathrm{W}'_i & = & \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \end{cases} & & & & & & & & \\
f'_i(X) & = & f_i(X) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1, 1) \rangle\} \circ f_k(X) & & & & & & & & \\
\mathbf{f}' & = & \mathbf{f} \restriction_{I' \cup J} & & & & & & & &
\end{array}
$$

$$
\begin{array}{rclcrclcrcl}
\mathcal{I}'' & = & \mathcal{I} \setminus \{\mathrm{I}_1, \mathrm{I}_2\} & & \mathcal{O}'' & = & \mathcal{O} \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & & \mathcal{J}'' & = & \mathcal{J} \\
\mathcal{V}'' & = & \{(\mathrm{V}_i, \mathrm{W}''_i, f''_i) \mid 3 \leq i \leq n_\mathrm{V}\} \text{ where} & & & & & & & & \\
\mathrm{W}''_i & = & \begin{cases} \mathrm{W}'_i & : \mathrm{O}_2 \notin \mathrm{W}'_i \\ (\mathrm{W}'_i \setminus \{\mathrm{O}_2\}) \cup \mathrm{W}'_l & : \mathrm{O}_2 \in \mathrm{W}'_i \end{cases} & & & & & & & & \\
f''_i(X) & = & f'_i(X) \cup f'_i(\mathrm{O}_2) \circ \{\langle (i_2, 1) \rangle\} \circ f'_l(X) & & & & & & & & \\
\mathbf{f}'' & = & \mathbf{f} \restriction_{I'' \cup J} & & & & & & & &
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{W}''_i & = & \begin{cases} \underline{\mathrm{W}'_i} & : \underline{\mathrm{O}_2 \notin \mathrm{W}'_i} \\ (\underline{\mathrm{W}'_i} \setminus \{\mathrm{O}_2\}) \cup \mathrm{W}'_l & : \underline{\mathrm{O}_2 \in \mathrm{W}'_i} \end{cases} \\
& = & \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \cup \mathrm{W}_k \\ (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \underline{\mathrm{W}'_l} & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k) \setminus \{\mathrm{O}_2\}) \cup \underline{\mathrm{W}'_l} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \cup \mathrm{W}_k \end{cases} \\
& = & \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \cup \mathrm{W}_k \\ (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \mathrm{W}_l & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \wedge \mathrm{O}_1 \notin \mathrm{W}_l \\ (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup (\mathrm{W}_l \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \underline{\mathrm{O}_1 \notin \mathrm{W}_i} \wedge \mathrm{O}_2 \in \mathrm{W}_i \wedge \underline{\mathrm{O}_1 \in \mathrm{W}_l} \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k) \setminus \{\mathrm{O}_2\}) \cup \mathrm{W}_l & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \cup \mathrm{W}_k \wedge \underline{\mathrm{O}_1 \notin \mathrm{W}_l} \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k) \setminus \{\mathrm{O}_2\}) \cup (\mathrm{W}_l \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \cup \mathrm{W}_k \wedge \underline{\mathrm{O}_1 \in \mathrm{W}_l} \end{cases}
\end{array}
$$

With (3) and $\mathrm{O}_1 \in \mathrm{W}_l$ we obtain $\mathrm{O}_2 \notin \mathrm{W}_k$. The following holds:

$$
\begin{array}{l}
\mathrm{O}_1 \notin \mathrm{W}_i \wedge \overbrace{\mathrm{O}_1 \notin \mathrm{W}_k}^{(1)} \wedge \overbrace{\mathrm{O}_2 \notin \mathrm{W}_k}^{(2)} \wedge \mathrm{O}_2 \notin \mathrm{W}_l \Rightarrow \\
\qquad (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup (\mathrm{W}_l \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k = (\mathrm{W}_i \cup \mathrm{W}_k \cup \mathrm{W}_l) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} \\
\mathrm{O}_1 \notin \mathrm{W}_k \wedge \mathrm{O}_1 \notin \mathrm{W}_l \wedge \mathrm{O}_2 \notin \mathrm{W}_l \Rightarrow \\
\qquad (((\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k) \setminus \{\mathrm{O}_2\}) \cup \mathrm{W}_l = (\mathrm{W}_i \cup \mathrm{W}_k \cup \mathrm{W}_l) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} \\
\mathrm{O}_1 \notin \mathrm{W}_k \wedge \mathrm{O}_2 \notin \mathrm{W}_k \wedge \mathrm{O}_2 \notin \mathrm{W}_l \Rightarrow \\
\quad (((\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k) \setminus \{\mathrm{O}_2\}) \cup (\mathrm{W}_l \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_l = (\mathrm{W}_i \cup \mathrm{W}_k \cup \mathrm{W}_l) \setminus \{\mathrm{O}_1, \mathrm{O}_2\}
\end{array}
$$

So the three last set terms can be simplified to the same term by using (1), (2) and (3). Therefore we obtain:

$$
\mathrm{W}''_i = \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_l) \setminus \{\mathrm{O}_2\} & : \mathrm{O}_1 \notin \mathrm{W}_i \cup \mathrm{W}_l \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_k) \setminus \{\mathrm{O}_1\} & : \mathrm{O}_2 \notin \mathrm{W}_i \cup \mathrm{W}_k \wedge \mathrm{O}_1 \in \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_k \cup \mathrm{W}_l) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \text{else} \end{cases}
$$

$$
\begin{aligned}
f_i''(X) &= f_i'(X) \cup f_i'(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l'(X) \\
&= f_i(X) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X) \cup \\
&\quad (f_i(\mathrm{O}_2) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(\mathrm{O}_2)) \circ \{\langle (i_2,1)\rangle\} \circ \\
&\quad (f_l(X) \cup f_l(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X)) \\
&= f_i(X) \cup \\
&\quad f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X) \cup \\
&\quad f_i(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l(X) \cup \\
&\quad f_i(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X) \cup \\
&\quad f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l(X) \cup \\
&\quad f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ \underline{f_k(\mathrm{O}_2)} \circ \{\langle (i_2,1)\rangle\} \circ \underline{f_l(\mathrm{O}_1)} \circ \{\langle (i_1,1)\rangle\} \circ f_k(X)
\end{aligned}
$$

Using (3) we obtain $f_k(\mathrm{O}_2) = \emptyset \vee f_l(\mathrm{O}_1) = \emptyset$ and therefore the following holds:

$$
\begin{aligned}
f_i''(X) &= f_i(X) \cup \\
&\quad f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X) \cup \\
&\quad f_i(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l(X) \cup \\
&\quad f_i(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X) \cup \\
&\quad f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(\mathrm{O}_2) \circ \{\langle (i_2,1)\rangle\} \circ f_l(X) \cup
\end{aligned}
$$

- First $\mathrm{O}_2 \to \mathrm{I}_2$, then $\mathrm{O}_1 \to \mathrm{I}_1$

Because of symmetric reasons we obtain the component that results in this case by determining the dual component of the one calculated above. It is obvious that these components are equal.

Case 2 I_1 independent and I_2 dependent. Then $k \neq l, \mathrm{V}_k = \{\mathrm{I}_1\}, \mathrm{I}_2 \in \mathrm{V}_l$ holds.

- First $\mathrm{O}_1 \to \mathrm{I}_1$, then $\mathrm{O}_2 \to \mathrm{I}_2$

$$
\begin{aligned}
\mathcal{I}' &= \mathcal{I} \setminus \{\mathrm{I}_1\} \qquad \mathcal{O}' = \mathcal{O} \setminus \{\mathrm{O}_1\} \qquad \mathcal{J}' = \mathcal{J} \\
\mathcal{V}' &= \{(\mathrm{V}_i, \mathrm{W}_i', f_i') \mid 2 \leq i \leq n_{\mathrm{V}}\} \text{ where} \\
\mathrm{W}_i' &= \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \end{cases} \\
f_i'(X) &= f_i(X) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1,1)\rangle\} \circ f_k(X) \\
\mathbf{f}' &= \mathbf{f} \restriction_{I' \cup J}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}'' &= \mathcal{I} \setminus \{\mathrm{I}_1, \mathrm{I}_2\} \qquad \mathcal{O}'' = \mathcal{O} \setminus \{\mathrm{O}_1, \mathrm{O}_2\} \qquad \mathcal{J}'' = \mathcal{J} \cup \{\mathrm{I}_2\} \\
\mathcal{V}'' &= \{(\mathrm{V}_i, \mathrm{W}_i'', f_i'') \mid 2 \leq i \leq n_{\mathrm{V}}\} \text{ where} \\
\mathrm{W}_i'' &= \begin{cases} \mathrm{W}_i' & : \mathrm{O}_2 \notin \mathrm{W}_i' \\ (\mathrm{W}_i' \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_2 \in \mathrm{W}_i' \end{cases} \\
f_i''(X) &= \begin{cases} f_i'(X) & : X \neq \mathrm{I}_2 \\ f_i'(\mathrm{O}_2) & : X = \mathrm{I}_2 \end{cases} \\
\mathbf{f}'' &= \mathbf{f} \restriction_{I'' \cup J}
\end{aligned}
$$

$$\mathrm{W}_i'' = \begin{cases} \mathrm{W}_i' & : \mathrm{O}_2 \notin \mathrm{W}_i' \\ (\mathrm{W}_i' \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_2 \in \mathrm{W}_i' \end{cases}$$

$$= \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k \\ (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k) \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k \end{cases}$$

Using (1) and set theoretical simplifications we obtain:

$$\mathrm{W}_i'' = \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_k) \setminus \{\mathrm{O}_1\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \cup \mathrm{W}_k \\ (\mathrm{W}_i \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_2\} & : \mathrm{O}_1 \notin \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_k \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \cup \mathrm{W}_k \end{cases}$$

$$f_i''(X) = \begin{cases} f_i(X) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1, 1) \rangle\} \circ f_k(X) & : X \neq \mathrm{I}_1 \\ f_i(\mathrm{O}_2) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1, 1) \rangle\} \circ f_k(\mathrm{O}_2) & : X = \mathrm{I}_1 \end{cases}$$

- First $\mathrm{O}_2 \to \mathrm{I}_2$, then $\mathrm{O}_1 \to \mathrm{I}_1$

$$\mathcal{I}^* = \mathcal{I} \setminus \{\mathrm{I}_2\} \qquad \mathcal{O}^* = \mathcal{O} \setminus \{\mathrm{O}_2\} \qquad \mathcal{J}^* = \mathcal{J} \cup \{\mathrm{I}_2\}$$

$$\mathcal{V}^* = \{(\mathrm{V}_i, \mathrm{W}_i^*, f_i^*) \mid 1 \leq i \leq n_{\mathrm{V}}\} \text{ where}$$

$$\mathrm{W}_i^* = \begin{cases} \mathrm{W}_i & : \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_2 \in \mathrm{W}_i \end{cases}$$

$$f_i^*(X) = \begin{cases} f_i(X) & : X \neq \mathrm{I}_2 \\ f_i(\mathrm{O}_2) & : X = \mathrm{I}_2 \end{cases}$$

$$\mathbf{f}^* = \mathbf{f} \restriction_{I^* \cup J}$$

$$\mathcal{I}^{**} = \mathcal{I} \setminus \{\mathrm{I}_1, \mathrm{I}_2\} \qquad \mathcal{O}^{**} = \mathcal{O} \setminus \{\mathrm{O}_1, \mathrm{O}_2\} \qquad \mathcal{J}^{**} = \mathcal{J}$$

$$\mathcal{V}^{**} = \{(\mathrm{V}_i, \mathrm{W}_i^{**}, f_i^{**}) \mid 2 \leq i \leq n_{\mathrm{V}}\} \text{ where}$$

$$\mathrm{W}_i^{**} = \begin{cases} \mathrm{W}_i^* & : \mathrm{O}_1 \notin \mathrm{W}_i^* \\ (\mathrm{W}_i^* \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k^* & : \mathrm{O}_1 \in \mathrm{W}_i^* \end{cases}$$

$$f_i^{**}(X) = f_i^*(X) \cup f_i^*(\mathrm{O}_1) \circ \{\langle (i_1, 1) \rangle\} \circ f_k^*(X)$$

$$\mathbf{f}^{**} = \mathbf{f} \restriction_{I^{**} \cup J}$$

$$
\begin{aligned}
\mathrm{W}_i^{**} &= \begin{cases} \mathrm{W}_i^* & : \mathrm{O}_1 \notin \mathrm{W}_i^* \\ (\mathrm{W}_i^* \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k^* & : \mathrm{O}_1 \in \mathrm{W}_i^* \end{cases} \\
&= \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_1 \notin (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k^* & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k^* & : \mathrm{O}_1 \in (\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} \wedge \mathrm{O}_2 \in \mathrm{W}_i \end{cases} \\
&= \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_2\}) & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_k \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup (\mathrm{W}_k \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \underline{\mathrm{O}_2 \notin \mathrm{W}_i} \wedge \mathrm{O}_2 \in \mathrm{W}_k \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1\}) \cup \mathrm{W}_k & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \wedge \underline{\mathrm{O}_2 \notin \mathrm{W}_k} \\ (((\mathrm{W}_i \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1\}) \cup (\mathrm{W}_k \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_k \end{cases} \\
&= \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_2\} & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_k) \setminus \{\mathrm{O}_1\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \cup \mathrm{W}_k \\ (\mathrm{W}_i \cup \mathrm{W}_k \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_k \\ (\mathrm{W}_i \cup \mathrm{W}_k \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_k \\ (\mathrm{W}_i \cup \mathrm{W}_k \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_k \end{cases}
\end{aligned}
$$

The last three conditions can be combined to one condition and we obtain:

$$
\mathrm{W}_i^{**} = \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_2\} & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \cup \mathrm{W}_k) \setminus \{\mathrm{O}_1\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \cup \mathrm{W}_k \\ (\mathrm{W}_i \cup \mathrm{W}_k \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \cup \mathrm{W}_k \end{cases}
$$

$$
f_i^{**}(X) = \begin{cases} f_i(X) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1, 1) \rangle\} \circ f_k(X) & : X \neq \mathrm{I}_1 \\ f_i(\mathrm{O}_1) \cup f_i(\mathrm{O}_1) \circ \{\langle (i_1, 1) \rangle\} \circ f_k(\mathrm{O}_1) & : X = \mathrm{I}_1 \end{cases}
$$

Again the following holds: $\mathcal{I}'' = \mathcal{I}^{**}, \mathcal{O}'' = \mathcal{O}^{**}, \mathcal{J}'' = \mathcal{J}^{**}, \mathrm{W}_i'' = \mathrm{W}_i^{**}, f_i''(X) = f_i^{**}(X), \mathbf{f}'' = \mathbf{f}^{**}$

Case 3 I_1 and I_2 are dependent. The $\mathrm{I}_1 \in \mathrm{V}_k, \mathrm{I}_2 \in \mathrm{V}_l$ holds.
- First $\mathrm{O}_1 \to \mathrm{I}_1$, then $\mathrm{O}_2 \to \mathrm{I}_2$

$$
\begin{aligned}
\mathcal{I}' &= \mathcal{I} \setminus \{\mathrm{I}_1\} \qquad \mathcal{O}' = \mathcal{O} \setminus \{\mathrm{O}_1\} \qquad \mathcal{J}' = \mathcal{J} \\
\mathcal{V}' &= \{(\mathrm{V}_i, \mathrm{W}_i', f_i') \mid 1 \leq i \leq n_{\mathrm{V}}\} \text{ where} \\
\mathrm{W}_i' &= \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \\ (\mathrm{W}_i \setminus \{\mathrm{O}_1\}) \cup \{\mathrm{I}_1\} & : \mathrm{O}_1 \in \mathrm{W}_i \end{cases} \\
f_i'(X) &= \begin{cases} f_i(X) & : X \neq \mathrm{I}_1 \\ f_i(\mathrm{O}_1) & : X = \mathrm{I}_1 \end{cases} \\
\mathbf{f}' &= \mathbf{f}
\end{aligned}
$$

$$
\begin{array}{rcl}
\mathcal{I}'' & = & \mathcal{I} \setminus \{\mathrm{I}_1, \mathrm{I}_2\} \qquad \mathcal{O}'' = \mathcal{O} \setminus \{\mathrm{O}_1, \mathrm{O}_2\} \qquad \mathcal{J}'' = \mathcal{J} \\
\mathcal{V}'' & = & \{(\mathrm{V}_i, \mathrm{W}_i'', f_i'') \mid 1 \leq i \leq n_{\mathrm{V}}\} \text{ where} \\
\mathrm{W}_i'' & = & \begin{cases} \mathrm{W}_i' & : \mathrm{O}_2 \notin \mathrm{W}_i' \\ (\mathrm{W}_i' \setminus \{\mathrm{O}_2\}) \cup \{\mathrm{I}_2\} & : \mathrm{O}_2 \in \mathrm{W}_i' \end{cases} \\
 & = & \begin{cases} \mathrm{W}_i & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \{\mathrm{I}_1\}) \setminus \{\mathrm{O}_1\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \notin \mathrm{W}_i \\ (\mathrm{W}_i \cup \{\mathrm{I}_2\}) \setminus \{\mathrm{O}_2\} & : \mathrm{O}_1 \notin \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \\ (\mathrm{W}_i \cup \{\mathrm{I}_1, \mathrm{I}_2\}) \setminus \{\mathrm{O}_1, \mathrm{O}_2\} & : \mathrm{O}_1 \in \mathrm{W}_i \wedge \mathrm{O}_2 \in \mathrm{W}_i \end{cases} \\
f_i''(X) & = & \begin{cases} f_i'(X) & : X \neq \mathrm{I}_2 \\ f_i'(\mathrm{O}_2) & : X = \mathrm{I}_2 \end{cases} \\
 & = & \begin{cases} f_i(X) & : X \neq \mathrm{I}_1 \wedge X \neq \mathrm{I}_2 \\ f_i(\mathrm{O}_1) & : X = \mathrm{I}_1 \\ f_i(\mathrm{O}_2) & : X = \mathrm{I}_2 \end{cases} \\
\mathbf{f}'' & = & \mathbf{f}
\end{array}
$$

- First $\mathrm{O}_2 \to \mathrm{I}_2$, then $\mathrm{O}_1 \to \mathrm{I}_1$

Because of symmetric reasons we obtain the component that results in this case by determining the dual component of the one calculated above. It is obvious that these components are equal.

□

The following lemma assures that there are no loops possible in components that are build out of ground components.

3.7 Lemma
For a component $C := (\mathcal{I}, \mathcal{O}, \mathcal{J}; \mathcal{V}, \mathbf{f})$ resulting from the composition of ground components the following holds:

$$X \in \mathrm{pre}(Y) \Longrightarrow Y \notin \mathrm{pre}(X)$$

3.3 Normal form

We now introduce three simplification rules and define a normal form of sequences and B-terms based on these rules. Afterwards we show that each sequence has a unique normal form.

3.8 Definition (Simplification rules)
For $\sigma, \tau \in \mathbb{S}_{\langle\rangle}$ and $x \in \{n \mid n \in \mathbb{N} \wedge n > 1\}$ we define the following simplifications on sequences of $\mathbb{S}$:

$$\frac{\sigma \circ \langle (a, bx)(cx, d) \rangle \circ \tau}{\sigma \circ \langle (a, b)(c, d) \rangle \circ \tau}\ (\mathrm{div})$$

$$\frac{\sigma \circ \langle (a, 1)(c, d) \rangle \circ \tau}{\sigma \circ \langle (ac, d) \rangle \circ \tau}\ (1\mathrm{l}) \qquad \frac{\sigma \circ \langle (a, b)(1, d) \rangle \circ \tau}{\sigma \circ \langle (a, bd) \rangle \circ \tau}\ (1\mathrm{r})$$

If a sequence τ is obtained by applying one of the rules above on a sequent σ we call this a 1-step-reduction and write $\sigma \rhd_1 \tau$.

We say that a sequence σ can be reduced to a sequence τ ($\sigma \rhd \tau$), if there exists a finite number of sequences $\rho_0, \rho_1, \ldots, \rho_n$, so that the following holds:

1. $\sigma = \rho_0$ and $\tau = \rho_n$,

2. $\rho_i \rhd_1 \rho_{i+1}$ for all $0 \leq i < n$.

3.9 Definition (Normal form)

1. A sequence $\sigma \in \mathbb{S}$ is in *normal form*, if no more simplification rule can be applied to it.
2. If $\sigma \rhd \tau$ holds and τ is in normal form then τ is called a *normal form of* σ.

A B-term

$$t := \sum_{A \in \mathrm{V}} i_A A \to \sum_{B \in \mathrm{W}} \sum_{\sigma \in f(B)} \sigma B$$

is said to be in *normal form*, if for all $B \in \mathrm{W}$ holds that each sequence $\sigma \in f(B)$ is in normal form.

3.10 Definition (Diamond property)

Let R be a Relation on $\mathbb{S} \times \mathbb{S}$. R is said to possess the diamond property, written $\mathrm{R} \models \diamond$, if for all sequences $\sigma, \tau_1, \tau_2 \in \mathbb{S}$ the following holds:

$$\sigma \mathrm{R} \tau_1 \quad \text{and} \quad \sigma \mathrm{R} \tau_2 \qquad \Rightarrow \qquad \exists \tau \in \mathbb{S} (\tau_1 \mathrm{R} \tau \text{ and } \tau_2 \mathrm{R} \tau)$$

3.11 Lemma

Let R be a Relation on $\mathbb{S} \times \mathbb{S}$ and let $\mathrm{R}*$ be its transitive closure. Then the following holds:

$$\mathrm{R} \models \diamond \quad \to \quad \mathrm{R}* \models \diamond$$

3.12 Definition ($\geq_1$)

Let $a, b, c, d, x \in \mathbb{N}^+, \tau_1, \tau_2 \in \mathbb{S}_{\langle\rangle}$. We define a relation $\geq_1 \subset \mathbb{S} \times \mathbb{S}$ as follows:

$$\begin{aligned} \tau_1 \circ \langle (a, bx)(cx, d) \rangle \circ \tau_2 \quad &\geq_1 \quad \tau_1 \circ \langle (a, b)(c, d) \rangle \circ \tau_2 \\ \tau_1 \circ \langle (a, 1)(c, d) \rangle \circ \tau_2 \quad &\geq_1 \quad \tau_1 \circ \langle (ac, d) \rangle \circ \tau_2 \\ \tau_1 \circ \langle (a, b)(1, d) \rangle \circ \tau_2 \quad &\geq_1 \quad \tau_1 \circ \langle (a, bd) \rangle \circ \tau_2 \end{aligned}$$

3.13 Lemma

The relation $\rhd$ is the transitive closure of the relation $\geq_1$

⌜

Obviously for all $\sigma, \tau \in \mathbb{S}$ holds:

1. $\sigma \rhd_1 \tau \quad \Rightarrow \quad \sigma \geq_1 \tau$
2. $\sigma \geq_1 \tau \quad \Rightarrow \quad \sigma \rhd \tau$

The assertion results from the fact that $\rhd$ is the reflexive and transitive closure of $\rhd_1$.

□

3.14 Lemma

let $\sigma, \tau, \tau' \in \mathbb{S}$ and $\tau \geq_1 \tau'$. The the following holds:

$$\sigma \circ \tau \geq_1 \sigma \circ \tau' \qquad \tau \circ \sigma \geq_1 \tau' \circ \sigma$$

3.15 Lemma

Let $\sigma_1, \sigma_2, \sigma_2' \in \mathbb{S}$ and $\sigma_1 \circ \sigma_2 \geq_1 \sigma_1 \circ \sigma_2'$. Then the following holds:

$$\sigma_2 \geq_1 \sigma_2'$$

3.16 Lemma
The relation $\geq_1$ possesses the diamond property, i.e. $\geq_1 \models \diamond$ holds.

⌜

With induction over the definition of $\sigma \geq_1 \tau_1$ we show that for all τ_2 the following holds:

$$\sigma \geq_1 \tau_2 \quad \rightarrow \quad \exists \tau (\tau_1 \geq_1 \tau \text{ and } \tau_2 \geq_1 \tau).$$

Case 1

$\sigma = \sigma_1 \circ \langle (a, bx)(cx, d) \rangle \circ \sigma_2$ where $x \in \mathbb{N}^+$
$\tau_1 = \sigma_1 \circ \langle (a, b)(c, d) \rangle \circ \sigma_2$

1. $\tau_2 = \sigma_1 \circ \langle (a, b')(c', d) \rangle \circ \sigma_2$ where $bx = b'y, cx = c'y$ for a $y \in \mathbb{N}^+$
   Let $z = \mathrm{lcm}(x, y)$
   Set $\tau = \sigma_1 \circ \langle (a, \frac{bx}{z})(\frac{cx}{z}, d) \rangle \circ \sigma_2$
2. $\tau_2 = \sigma_3 \circ \langle (a, bx)(cx, d) \rangle \circ \sigma_2$
   According to lemma 3.15 $\sigma_1 \geq_1 \sigma_3$ holds.
   Set $\tau = \sigma_3 \circ \langle (a, b)(c, d) \rangle \circ \sigma_2$ (lemma 3.14)
3. $\tau_2 = \sigma_1 \circ \langle (a, bx)(cx, d) \rangle \circ \sigma_3$ analogical
4. $\sigma = \sigma_3 \circ \langle (e, fy)(a, bx)(cx, d) \rangle \circ \sigma_2$ where $a | y$
   $\tau_1 = \sigma_3 \circ \langle (e, fy)(a, b)(c, d) \rangle \circ \sigma_2$
   $\tau_2 = \sigma_3 \circ \langle (e, f)(\frac{a}{y}, bx)(cx, d) \rangle \circ \sigma_2$
   Set $\tau = \sigma_3 \circ \langle (e, f)(\frac{a}{y}, b)(c, d) \rangle \circ \sigma_2$
5. $\sigma = \sigma_1 \circ \langle (a, bx)(cx, d)(ey, f) \rangle \circ \sigma_3$ where $d | y$
   $\tau_1 = \sigma_1 \circ \langle (a, b)(c, d)(ey, f) \rangle \circ \sigma_3$
   $\tau_2 = \sigma_1 \circ \langle (a, bx)(cx, \frac{d}{y})(e, f) \rangle \circ \sigma_3$
   Set $\tau = \sigma_1 \circ \langle (a, b)(c, \frac{d}{y})(e, f) \rangle \circ \sigma_3$
6. $\sigma = \sigma_3 \circ \langle (e, 1)(a, bx)(cx, d) \rangle \circ \sigma_2$
   $\tau_1 = \sigma_3 \circ \langle (e, 1)(a, b)(c, d) \rangle \circ \sigma_2$
   $\tau_2 = \sigma_3 \circ \langle (ea, bx)(cx, d) \rangle \circ \sigma_2$
   Set $\tau = \sigma_3 \circ \langle (ea, b)(c, d) \rangle \circ \sigma_2$
7. $\sigma = \sigma_1 \circ \langle (a, bx)(cx, d)(1, f) \rangle \circ \sigma_3$
   $\tau_1 = \sigma_1 \circ \langle (a, b)(c, d)(1, f) \rangle \circ \sigma_3$
   $\tau_2 = \sigma_1 \circ \langle (a, bx)(cx, df) \rangle \circ \sigma_3$
   Set $\tau = \sigma_1 \circ \langle (a, b)(c, df) \rangle \circ \sigma_3$

Case 2

$\sigma = \sigma_1 \circ \langle (a, 1)(c, d) \rangle \circ \sigma_2$
$\tau_1 = \sigma_1 \circ \langle (ac, d) \rangle \circ \sigma_2$

1. $\tau_2 = \tau_1$
   Set $\tau = \tau_1$
2. $\tau_2 = \sigma_3 \circ \langle (a, 1)(c, d) \rangle \circ \sigma_2$
   According to lemma 3.15 $\sigma_1 \geq_1 \sigma_3$ holds.
   Set $\tau = \sigma_3 \circ \langle (ac, d) \rangle \circ \sigma_2$ (lemma 3.14)
3. $\tau_2 = \sigma_1 \circ \langle (a, 1)(c, d) \rangle \circ \sigma_3$ analogical
4. $\sigma = \sigma_3 \circ \langle (e, fy)(a, 1)(c, d) \rangle \circ \sigma_2$ where $a | y$
   $\tau_1 = \sigma_3 \circ \langle (e, fy)(ac, d) \rangle \circ \sigma_2$
   $\tau_2 = \sigma_3 \circ \langle (e, f)(\frac{a}{y}, 1)(c, d) \rangle \circ \sigma_2$
   Set $\tau = \sigma_3 \circ \langle (e, f)(\frac{ac}{y}, d) \rangle \circ \sigma_2$
5. $\sigma = \sigma_1 \circ \langle (a, 1)(c, d)(ey, f) \rangle \circ \sigma_3$ where $d | y$
   $\tau_1 = \sigma_1 \circ \langle (ac, d)(ey, f) \rangle \circ \sigma_3$
   $\tau_2 = \sigma_1 \circ \langle (a, 1)(c, \frac{d}{y})(e, f) \rangle \circ \sigma_3$
   Set $\tau = \sigma_1 \circ \langle (ac, \frac{d}{y})(e, f) \rangle \circ \sigma_3$

6. $\sigma = \sigma_3 \circ \langle (e,1)(a,1)(c,d) \rangle \circ \sigma_2$
   $\tau_1 = \sigma_3 \circ \langle (e,1)(ac,d) \rangle \circ \sigma_2$
   $\tau_2 = \sigma_3 \circ \langle (ae,1)(c,d) \rangle \circ \sigma_2$
   Set $\tau = \sigma_3 \circ \langle (ace,d) \rangle \circ \sigma_2$
7. $\sigma = \sigma_1 \circ \langle (a,1)(c,d)(1,f) \rangle \circ \sigma_3$
   $\tau_1 = \sigma_1 \circ \langle (ac,d)(1,f) \rangle \circ \sigma_3$
   $\tau_2 = \sigma_1 \circ \langle (a,1)(c,df) \rangle \circ \sigma_3$
   Set $\tau = \sigma_3 \circ \langle (ac,df) \rangle \circ \sigma_2$

Case 3
$\sigma = \sigma_1 \circ \langle (a,b)(1,d) \rangle \circ \sigma_2$
$\tau_1 = \sigma_1 \circ \langle (a,bd) \rangle \circ \sigma_2$
analogical to case 2.

□

3.17 Lemma
Each sequence $\sigma \in \mathbb{S}$ has a normal form.

⌜

Let $\sigma = \langle (x_1, y_1) \dots (x_n, y_n) \rangle$ and $p : \mathbb{S} \to \mathbb{N}$ the following function:

$$p(\sigma) = \prod_{i=1}^{n} x_i y_i.$$

We then define a relation $<$ on $\mathbb{S} \times \mathbb{S}$ as follows:

$$\sigma < \tau :\Longleftrightarrow \big(\mathrm{len}(\sigma) < \mathrm{len}(\tau) \wedge p(\sigma) \leq p(\tau) \big) \vee \big(\mathrm{len}(\sigma) = \mathrm{len}(\tau) \wedge p(\sigma) < p(\tau) \big)$$

The minimal element regarding $<$ is $\langle (1,1) \rangle$.

With each simplification step a smaller element is formed until we reach $\langle (1,1) \rangle$ which can not simplified anymore or another element that can not be simplified.

□

3.18 Proposition
Each sequence $\sigma \in \mathbb{S}$ possesses a unique normal form.

⌜

Follows directly from 3.11, 3.13 and 3.16.

□

3.19 Corollary
Each B-term posses a unique normal form.

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