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Abstract

We show that a uniform version of the limit axiom does not strengthen the theory UTN of universes, types, and names plus (non-uniform) limes.

1 Introduction

Systems of explicit mathematics were introduced by Feferman in [Fef75, Fef79]. They provide a logical account to constructive mathematics and functional programming. Moreover, they became very important for proof-theoretic analysis of subsystems of second order arithmetic and systems of Kripke-Platek set theory.

Universes are well-known from Martin-Löf's type theory. In the framework of explicit mathematics they are first introduced and studied by Feferman in [Fef82]. Later on, Marzetta has defined the theory UTN of *universes over types and names* in [Mar93, Mar94]. This theory is based on the theory EET of Feferman and Jäger [FJ96]. EET is formulated in a language with two sorts of objects, *operations* and *types*. In EET a naming relation $\Re$ allows to represent (extensional) types by (intensional) operations. In UTN the existence of universes is guaranteed by a so-called *limit axiom*. This axiom is taken from Kripke-Platek set theory illustrating the close relationship of universes and admissible sets (cf. [Jäg84]). The limit axiom postulates for each type the existence of a universe containing a name of this type. In fact it guarantees the existence of finitely many universes.

In this paper we focus on the formulation of this limit axiom, which is *non uniform*, i.e. there is no functional relation between the given type and (the name of) the universe above it. In the framework of explicit mathematics uniformity in type existence axioms is a very natural requirement, with respect to the proof-theoretic results as well as for the constructive justification, cf. [Gla93, Gla95]. In the following we show that a *uniform* formulation of the limit axiom, which provides for a given name of a type a name of a universe above it, does not increase the proof-theoretic strength in the presence of type induction. For the proof we follow the presentation of the proof theoretic analysis of UTN with non uniform limit in [MS9x].

2 UTN

The language L_{U} of UTN is a two sorted language comprising individual variables $(a, b, c, x, y, z, f, g, \ldots)$; individual constants: k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and projection), $0, \mathsf{s_N}, \mathsf{p_N}, \mathsf{d_N}$ (zero, successor, predecessor, definition by numerical cases), j (join), c_n (comprehension) for every $n \in I\!N$; type variables $A, B, C, X, Y, Z, \ldots$; a binary function symbol $\cdot$ for application of individuals.

The relation symbols of L_{U} include equality for individuals and types, $\downarrow$ (definedness) and N (natural numbers) on individuals, and U (universes) on types. There are two binary relations between individuals and types: $\in$ (membership) and $\Re$ (naming). Since we will work in a Tait calculus there is also a negative predicate symbol $\bar{P}$ for every positive relation P.

Individual terms $(r, s, t, \ldots)$ are build form individual variables and constants by application. Type terms $(R, S, T, \ldots)$ are just the type variables.

We write $t \cdot s$ instead of $(t \cdot s)$ with the convention of association to the left, and (t, s) for $\mathsf{p}\, t\, s$.

Formulae are build as usual by $\wedge, \vee, \exists$, and $\forall$. We define negation starting from the negative predicate symbols using the rules of De Morgan. As abbreviation we set:

$$T \mathrel{\dot\in} S \quad :\Leftrightarrow \quad \exists x.\Re(x, T) \wedge x \in S.$$

Elementary formulae are formulae without type quantifiers and without occurrences of the predicates $\Re$ and U, nor of their complementary predicates $\bar{\Re}, \bar{\mathsf{U}}$.

For the metamathematical investigations UTN is formulated as a Tait calculus, but for readability we omit the side formulae in the following.

The logic of UTN is the classical logic of partial terms [Bee85] for the individuals, and classical logic with equality for the types. The non-logical axioms contain those of EET [FJ96], which comprises partial combinatory logic, pairing and projection, natural numbers, definition by numerical cases, extensionality for sets, ontological axioms dealing with the naming relation, elementary comprehension, and join. Moreover we have two groups of axioms for universes: closure properties and ontological axioms. Finally, we have in UTN induction for types.

In the axioms for comprehension we use m as a Gödel number of the elementary formula $\varphi(x, \vec{a}, \vec{B})$ which does not contain the variables C and Z. In the rules for join we will use the abbreviation $Z \subset \Sigma(S, t)$ for the formula

$$\forall z. z \in Z \to z = (\mathsf{p}_0\, z, \mathsf{p}_1\, z) \wedge \mathsf{p}_0\, z \in S \wedge \exists X.\Re(t\,(\mathsf{p}_0\, z), X) \wedge \mathsf{p}_1\, z \in X$$

and $Z \supset \Sigma(S, t)$ for the formula

$$\forall z. z = (\mathsf{p}_0\, z, \mathsf{p}_1\, z) \wedge \mathsf{p}_0\, z \in S \wedge (\forall X.\Re(t\,(\mathsf{p}_0\, z), X) \to \mathsf{p}_1\, z \in X) \to z \in Z.$$

I. **Partial combinatory algebra**

(1) $\mathsf{k}\, a\, b = a$,

(2) $\mathsf{s}\, a\, b \downarrow \wedge\ \mathsf{s}\, a\, b\, c \simeq a\, c\, (b\, c)$.

II. **Pairing and projection**

(3) $\mathsf{p}_0\, (a, b) = a \wedge \mathsf{p}_1\, (a, b) = b$.

III. **Natural numbers**

(4) $\mathsf{N}(0) \wedge \forall x. \mathsf{N}(x) \to \mathsf{N}(\mathsf{s_N}\, x)$,

(5) $\forall x. x \in N \to \neg \mathsf{s_N}\, x = 0 \wedge \mathsf{p_N}\, (\mathsf{s_N}\, x) = x$,

(6) $\forall x. x \in N \wedge \neg x = 0 \to \mathsf{N}(\mathsf{p_N}\, x) \wedge \mathsf{s_N}\, (\mathsf{p_N}\, x) = x$.

IV. **Definition by numerical cases**

(7) $\mathsf{N}(a) \wedge \mathsf{N}(b) \wedge a = b \to \mathsf{d_N}\, c\, d\, a\, b = c$,

(8) $\mathsf{N}(a) \wedge \mathsf{N}(b) \wedge \neg a = b \to \mathsf{d_N}\, c\, d\, a\, b = d$.

V. **Extensionality**

(9) $(\forall x. x \in A \leftrightarrow x \in B) \to A = B$.

VI. **Ontological axioms**

(10) $\Re(a, B) \wedge \Re(a, C) \to B = C$,

(11) $\exists x. \Re(x, A)$.

VII. **Elementary comprehension**

(12) $\exists Z. \forall x. x \in Z \leftrightarrow \varphi(x, \vec{a}, \vec{B})$,

(13) $\neg \Re(\vec{b}, \vec{B}), \neg \forall x. x \in C \leftrightarrow \varphi(x, \vec{a}, \vec{B}), \Re(\mathsf{c}_m\, \vec{a}\, \vec{b}, C)$.

VIII. **Join**

$$(J_1)\ \frac{\Gamma, t \downarrow \wedge\ \Re(s, S) \wedge \forall x. x \in S \to \exists X. \Re(t\, x, X)}{\Gamma, \exists Z. \Re(\mathsf{j}\, s\, t, Z) \wedge Z \subset \Sigma(S, t)}$$

$$(J_2)\ \frac{\Gamma, t \downarrow \wedge\ \Re(s, S) \wedge \forall x. x \in S \to \exists X. \Re(t\, x, X)}{\Gamma, \exists Z. \Re(\mathsf{j}\, s\, t, Z) \wedge Z \supset \Sigma(S, t)}$$

IX. **Ontological axioms for the universes**

(14) $\mathsf{U}(A) \wedge b \in A \to \exists X. \Re(b, X)$,

(15) $\mathsf{U}(A) \wedge \mathsf{U}(B) \to A \,\dot{\in}\, B \vee A = B \vee B \,\dot{\in}\, A$,

(16) $\mathsf{U}(A) \wedge \mathsf{U}(B) \wedge A \,\dot{\in}\, B \to \forall x. x \in A \to x \in B$.

X. **Closure properties of universes**

(17) $\mathsf{U}(C) \wedge \vec{b} \in C \to \mathsf{c}_m\, \vec{a}\, \vec{b} \in C$,

(18) $\mathsf{U}(C) \wedge a \in C \wedge \Re(a, A) \wedge (\forall x. x \in A \to f\, x \in C) \to \mathsf{j}\, a\, f \in C$.

XI. **Induction axiom**

(19) $0 \in A \wedge (\forall x. \mathsf{N}(x) \wedge x \in A \to \mathsf{s_N}\, x \in A) \to \forall x. \mathsf{N}(x) \to x \in A$.

3 The non-uniform limes

The existence of universes is guaranteed by the *limes axiom*, which Marzetta defines in the following *non uniform* way:

(Lim) $\quad \exists Y. \mathsf{U}(Y) \wedge A \,\dot{\in}\, Y$.

The theory $\mathsf{UTN} + (\mathsf{Lim})$ is reducible to theories with finitely many universes. To show this, one defines the theories UTN_k, $k \in I\!N$, with k universes. The language of UTN_k is L_U extended by type constants D_i $(0 \leq i < k)$, the axiom are those of UTN plus the following:

XII. **Universes**

(20) $\mathsf{U}(\mathsf{D}_i)$, $\qquad (0 \leq i < k)$,

(21) $\mathsf{D}_i \,\dot{\in}\, \mathsf{D}_{i+1}$, $\qquad (0 \leq i < k-1)$.

The reduction of $\mathsf{UTN}+(\mathsf{Lim})$ to UTN_k is proved in [MS9x, Sec. 2.2] by an asymmetric interpretation.

Definition 1 [MS9x, Def. 1] An L_U formula is called $\Sigma+$ (respectively $\Pi-$), if it does not contain any universal (resp. existential) quantifier over types, nor any subformula of the form $\overline{\Re}(t, T)$ (resp. $\Re(t, T)$).

Since the chosen axiomatization of UTN has only $\Sigma+$ and $\Pi-$ formulae as principal formulae in the non logical axioms and rules, we get partial cut elmination up to $\Sigma+$ and $\Pi-$ formulae. With the following asymmetric interpretation one gets the embedding of $\mathsf{UTN} + (\mathsf{Lim})$ in UTN_k.

Definition 2 [MS9x, Def. 8] For natural number m and n and a L_U formula φ we define inductively $\varphi^{(m,n)}$:

1. $\Re(t, T)^{(m,n)} :\Leftrightarrow \Re(t, T) \wedge t \in \mathsf{D}_n$,
2. $\overline{\Re}(t, T)^{(m,n)} :\Leftrightarrow \neg(\Re(t, T) \wedge t \in \mathsf{D}_m)$,
3. $\varphi^{(m,n)} :\Leftrightarrow \varphi$ for all other atomic formulae of L_U,

4. $(\forall X.\varphi)^{(m,n)} :\Leftrightarrow \forall X \mathrel{\dot{\in}} \mathsf{D}_m.\varphi^{(m,n)}$,
5. $(\exists X.\varphi)^{(m,n)} :\Leftrightarrow \exists X \mathrel{\dot{\in}} \mathsf{D}_n.\varphi^{(m,n)}$,
6. homomorphically for propositional connectives and quantifiers ranging over individuals.

This definition is extend to finite sets Γ of L_{U} formulae in the straightword way.

Theorem 3 (Asymmetric interpretation) [MS9x, Th. 10] Let $\Gamma(\vec{x}, X_1, \ldots, X_i)$ be a finite set of $\Sigma+$ and $\Pi-$ formulae of L_{U}. Let l, m be natural numbers and $n := m + 2^l$. Assume $\mathsf{UTN} + (\mathsf{ULim}) \vdash^l_1 \Gamma$. Then

$$\mathsf{UTN}_k \vdash \neg X_1 \mathrel{\dot{\in}} \mathsf{D}_m, \ldots, \neg X_i \mathrel{\dot{\in}} \mathsf{D}_m, \Gamma(\vec{x}, X_1, \ldots, X_i)^{(m,n)}$$

holds for all $k > n$.

In a second step Marzetta and Strahm model UTN_k in the fixed point theories $\widehat{\mathrm{ID}\Omega}^r_{k+1}$. These theories extend the well-known theories $\widehat{\mathrm{ID}}_n$ of iterated fixed points by ordinals and comprise a suitable form of induction over the ordinals. The formal definition is given in [MS9x, Sec. 3.1].

Theorem 4 [MS9x, Th. 18] UTN_k can be embedded in $\widehat{\mathrm{ID}\Omega}^r_{k+1}$.

Here we give only a rough sketch of the proof, for details cf. [MS9x, Sec. 3.2]. First one chooses codes for the types:

- $\langle 0, m, \vec{a}, \vec{b}\rangle$ for the type obtained by elementary comprehension of a formula $\varphi(x, \vec{a}, \vec{B})$ with Gödel number m, if $\vec{b}$ are the codes for $\vec{B}$,
- $\langle 1, a, f\rangle$ for the join of f over a type coded by a,
- $\langle 2, i\rangle$ for the universe D_i.

The interpretations of type, the element, and inverse element relation, QT^k, $Q\epsilon^k$, and $Q\overline{\epsilon}^k$, respectively, are defined in UTN_k simultaneously as fixed points of appropriate operator forms. This operator forms using QT, $Q\epsilon$, and $Q\overline{\epsilon}$ as free relation symbols and the relations QT^i, $Q\epsilon^i$, and $Q\overline{\epsilon}^i$ of the embedding of the lower theories UTN_i, $i < k$.[1] They have four clauses each. The first ones inherit the elements of the lower relations QT^{k-1}, $Q\epsilon^{k-1}$, and $Q\overline{\epsilon}^{k-1}$, the second ones formalize the closure under elementary comprehension and the third ones closure under join, and finally the forth ones capture the universes. Only the latter are relevant for the difference of non uniform and uniform limit. It reads for $QT(y_0)$:

$$y_0 = \langle 2, k-1\rangle$$

[1]To avoid case distinctions we set $QT^m(x) :\equiv Q\epsilon^m(x, y) :\equiv Q\overline{\epsilon}^m(x, y) :\equiv 0 = 1$ for $m < 0$.

and for $Q\epsilon(x_0, y_0)$:

$$y_0 = \langle 2, k-1\rangle \wedge QT^{k-1}(x_0).$$

Defining $Q\overline{\epsilon}^k(x,y)$ in a corresponding way, the complementarity of $Q\epsilon^k(x,y)$ and $Q\overline{\epsilon}^k(x,y)$ under the assumption $QT^k(y)$ can be shown by induction on the ordinals in $\widehat{\mathrm{ID}\Omega}^r_{k+1}$. We interprete the naming relation as type equality "$a = b$", whose extensional interpretation is given by $QT^k(a) \wedge QT^k(b) \wedge \forall x.Q\epsilon^k(x,a) \leftrightarrow Q\epsilon^k(x,b)$. $\mathsf{U}(a)$ is translated by "$a = \langle 2,0\rangle$" $\vee \ldots \vee$ "$a = \langle 2, k-1\rangle$". Now the verification of the embedding becomes a straightforward calculation.

Since $\widehat{\mathrm{ID}\Omega}^r_{n+1}$ has the proof-theoretic ordinal γ_n [MS9x, Th. 27] one gets together with the result for the lower bound [MS9x, Th. 5]:

Theorem 5 [MS9x, Th. 30] $\mathsf{UTN} + (\mathsf{Lim})$ has the proof-theoretic ordinal Γ_0.

4 The uniform limes

Now we investigate the limes axiom in a uniform version where a name of the universe depends on the set for which a name x is given. For this aim we extend the language L_{U} by the new individual constant lim. In the usual Hilbert style presentation of explicit mathematics the uniform limit axiom reads

$$\forall x.(\exists Z.\Re(x,Z)) \to \exists Y.\mathsf{U}(Y) \wedge \Re(\mathsf{lim}\, x, Y) \wedge x \in Y.$$

In the Tait calculus we introduce it as an (equivalent) rule with a $\Sigma+$ formula as principal formula.

$$(\mathsf{ULim}) \quad \frac{\Gamma, \exists Z.\Re(t,Z)}{\Gamma, \exists Y.\mathsf{U}(Y) \wedge \Re(\mathsf{lim}\, t, Y) \wedge x \in Y}$$

To show that $\mathsf{UTN} + (\mathsf{ULim})$ is not stronger than $\mathsf{UTN} + (\mathsf{Lim})$ we can follow the lines of the proof for the upper bound in [MS9x]. First we extend the intermediate theories UTN_k.

$\mathsf{UTN}^\star_k$ is the theory UTN_k with the additional constant lim plus the following axiom:

XII. **Universes**

(22) $x \in \mathsf{D}_i \to \exists Y.\mathsf{U}(Y) \wedge \Re(\mathsf{lim}\, x, Y) \wedge x \in Y \wedge \mathsf{lim}\, x \in \mathsf{D}_{i+1}, (0 \leq i < k-1)$.

Since the principal formula of (ULim) is $\Sigma+$, we can still prove partial cut elimination up to $\Sigma+$ and $\Pi-$ formulae. Using the same asymmetric interpretation as for $\mathsf{UTN}+(\mathsf{Lim})$ we get also an embedding of $\mathsf{UTN} + (\mathsf{ULim})$ in $\mathsf{UTN}^\star_k$:

Theorem 6 (Asymmetric interpretation) Let $\Gamma(\vec{x}, X_1, \ldots, X_i)$ be a finite set of $\Sigma+$ and $\Pi-$ formulae. Let l, m be natural numbers and $n := m + 2^l$. Assume $\mathsf{UTN} + (\mathsf{ULim}) \vdash^l_1 \Gamma$. Then

$$\mathsf{UTN}^\star_k \vdash \neg X_1 \mathrel{\dot\in} \mathsf{D}_m, \ldots, \neg X_i \mathrel{\dot\in} \mathsf{D}_m, \Gamma(\vec{x}, X_1, \ldots, X_i)^{(m,n)}$$

holds for all $k > n$.

Proof: As the corresponding theorem for $\mathsf{UTN} + (\mathsf{Lim})$ we prove this theorem by induction on l. The only interesting case here is the uniform limes.
By the induction hypothesis we have for $l_0 < l$

$$\vdash^{l_0} \exists Z \dot{\in} \mathsf{D}_{m+2^{l_0}}.\Re(t, Z) \wedge t \in \mathsf{D}_{m+2^{l_0}}.$$

Axiom (22) yields $\exists Y.\mathsf{U}(Y) \wedge \Re(\mathsf{lim}\, t, Y) \wedge t \in Y \wedge \mathsf{lim}\, t \in \mathsf{D}_{m+2^{l_0}+1}$. With (21) and $l_0 < l$ we get $\mathsf{lim}\, t \in \mathsf{D}_{m+2^l}$ and have as desired conclusion

$$\vdash^{l} \exists Y \dot{\in} \mathsf{D}_{m+2^l}.\mathsf{U}(Y) \wedge \Re(\mathsf{lim}\, t, Y) \wedge \mathsf{lim}\, t \in \mathsf{D}_{m+2^l} \wedge t \in Y. \qquad \square$$

Now we give the modifications of the embedding of UTN_k in $\widehat{\mathrm{ID}\Omega}^r_{k+1}$ to verify the additional axiom of $\mathsf{UTN}^\star_k$.
Essentially we have to change the codes for the universes, using $\underbrace{\langle 2, \langle 2, \ldots, \langle 2}_{i \text{ times}}, n\rangle \cdots\rangle\rangle$ as code for the universe D_i (instead of $\langle 2, i\rangle$); here n is the code for the type of natural numbers.
So lim can be interpreted by the function which maps x to $\langle 2, x\rangle$.
For $QT(y_0)$ we replace the forth clause $y_0 = \langle 2, k-1\rangle$ by

$$\exists y.QT^{k-1}(y) \wedge \neg QT^{k-2}(y) \wedge y_0 = \langle 2, y\rangle. \qquad (\star)$$

In the definition of $Q\epsilon(x_0, y_0)$ the last clause $y_0 = \langle 2, k-1\rangle \wedge QT^{k-1}(x_0)$ changes to

$$\exists y.QT^{k-1}(y) \wedge \neg QT^{k-2}(y) \wedge y_0 = \langle 2, y\rangle \wedge QT^{k-1}(x_0). \qquad (\star\star)$$

$Q\overline{\epsilon}(x_0, y_0)$ will be adapted in the corresponding way.
The complementarity of $Q\epsilon^k(x, y)$ and $Q\overline{\epsilon}^k(x, y)$ under the assumption $QT^k(y)$ can be shown as for UTN_k. Also naming relation and type equality is interpreted as for UTN_k. But the translation of $\mathsf{U}(a)$ changes to $QT^k(a) \wedge \exists x.$"$a = \langle 2, x\rangle$"

Theorem 7 $\mathsf{UTN}^\star_k$ can be embedded in $\widehat{\mathrm{ID}\Omega}^r_{k+1}$.

Proof: In addition to the proof of Marzetta and Strahm we have to check the axioms for the universes, i.e. the groups IX (ontological axioms for the universes), X (closure properties of universes), and XII (universes).
First we note, that universes are interpreted by codes $\langle 2, a\rangle$ where a is a code of a type. If $st(a)$ is the minimal number j such that a is a code of a type at stage j, i.e. we have $QT^j(a)$, it follows from $(\star)$ that $st(\langle 2, a\rangle) = st(a) + 1$. By $(\star\star)$ it follows that the elements of this universe are exactly the codes of all types from the previous stage.
From these considerations, the verification of the ontological axioms IX as well as the closure conditions X for universes is straightforward.

The axiom XII.(20), expressing that every D_i for $i < k$ is a universe, follows by (meta-)induction on i.

Since the code of D_i, $i < k-1$, belongs to QT^i, it is also an element of D_{i+1}.

To verify the last axiom of UTN_k^* we set $j := st(x)$. From the premise $x \in \mathsf{D}_i$, $i < k$, it follows that $j < i$. So x is an element of $\langle 2, x\rangle = \mathsf{lim}\, x$ at stage $j+1$. Moreover $\mathsf{lim}\, x$ is a universe at stage $j+1$. So this universe is an element of D_{j+2}, and in particular of D_{i+1}. Since the naming relation was interpreted as extensional equality, $\mathsf{lim}\, x$ can be chosen as the witness Y. $\square$

As corollary we get

Corollary 8 $\mathsf{UTN} + (\mathsf{ULim})$ has the proof-theoretic ordinal Γ_0.

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