Fast Load Balancing in Cayley Graphs and in Circuits

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Abstract. We compare two load balancing techniques for Cayley graphs based on information and load exchange between neighboring vertices. In the first scheme, called natural diffusion, each vertex gives (or receives) a fixed part of the load difference to (from) its direct neighbors. In the second scheme, called Cayley diffusion, each vertex successively gives (or receives) a part of the load difference to (or from) direct neighbors incident to the edges labeled by the elements of the generator set of the Cayley graph. We prove that the convergence of the Cayley diffusion is faster than the natural diffusion, at least for some particular graphs (cube, circuit with an even number of vertices, graphs from the symmetric group). Furthermore we compute the fastest possible way to distribute load in a circuit using local load balancing strategies.
Topics covered. Theory of Parallel and Distributed Computation, Parallel Algorithms, Load Balancing, Cayley Graphs, Complexity.

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1 Introduction

To achieve high performance with a parallel computer it is imperative to balance the work-load of the processors. Various strategies have been proposed to solve this problem [Hać89]. We will focus here on parallel load balancing algorithms for interconnected multiprocessor networks based on local load exchange, i.e. on algorithm with the following structure:

Algorithm 1 (local load balancing strategy).

```
WHILE TRUE
  PAR i = 1 FOR processors
    exchange load with direct neighbors
```

Furthermore, we will assume that at any moment, each processor has only informations about the load of its direct neighbours.
There are many possibilities for local load exchange [ELZ86b], [ELZ86a], [MTS90], [Cyb89], etc. The *natural* concept is to give (or receive) a fixed part of the load difference to (from) all direct neighbors at the same time [Cyb89], [Boi90]. This strategy should be applied if the structure of the underlying network (graph) is unknown. In [Boi90] we have shown that this method is equivalent to a time discrete Poisson equation in a finite undirected graph. We also have shown how to choose the load amount to exchange for ensuring the convergence of the algorithm in any connected undirected graph.
Each load exchange step with one particular neighbor needs 2 communications. The first for getting the neighbor's load and the second for exchanging the load. In a multiprocessor system, communication may be cost intensive and excessive synchronization steps may result in poor efficiency.
In this paper we will show that it is possible, at least for some Cayley graphs, to achieve faster convergence by exchanging load in a round robin fashion with the neighbors.

2 Natural load balancing

In this section, we define the natural load balancing technique. Furthermore, we show how to choose the fixed amount of local load exchange to achieve the fastest possible convergence of the natural load balancing strategy in circuit graphs.

2.1 Natural load balancing

In this paper, a graph $G = (V, E)$ is a non directed, connected, and regular finite graph without loops. We will denote the adjacency matrix of G by A. Furthermore, we will denote the degree of G by k.

Definition 1 (Natural diffusion in regular graphs). Let G be a regular graph of degree k, and let $\alpha \in [0, 1]$ be a real number. Consider the symmetric stochastic matrix

$$P_\alpha = \alpha I + \frac{1-\alpha}{k} A \qquad (1)$$

The *natural diffusion* is defined by the matrix P_α, or more precisely by solving the equation

$$x_{t+1} = P_\alpha x_t = P_\alpha^t x_0 \quad (t \in I\!N) \qquad (2)$$

where the series of powers of P_α denote the successive load balancing steps.

Let $i \in V$ be a vertex of G. We denote by N_i the set of neighboring vertices of i. The natural diffusion corresponds to following parallel algorithm:

Algorithm 2 (Natural load balancing algorithm).

```
WHILE TRUE
  PAR i = 1 FOR processors
    PAR j ∈ N_i
      give (1-α)/k times the load difference to direct neighbor j
```

Algorithm 2 always converges towards the uniform distribution, provided $\alpha \in]0,1[$. It may converge if $\alpha = 0$ (see [Boi90]). Given a graph G it is always possible to find $\alpha \in [0,1]$ such that the convergence of algorithm 2 is the fastest as possible.

Note 2. In practice, the load of a processor may consist in a finite sum of time complexities of (atomic) processes. If the number of atomic processes is much larger than the number of processors, we may consider the load of a processor as a real number. Tests have shown that algorithm 2 behaves well, if the processes have discrete loads, provided there are many more processes than processors [BBK91].

2.2 Optimal natural load balancing in circuits

In this section, we show how to chose α such that the convergence of the natural diffusion scheme is optimal in a circuit. Note that the method used here is the same for any regular graph.
Let A denote the adjacency matrix of the undirected circuit with n vertices ($n \geq 3$), and let $\alpha \in [0,1]$ be a real number. Recall that the associated stochastic matrix is

$$P_\alpha = \alpha I + \frac{1-\alpha}{2} A \tag{3}$$

The convergence speed of this Markov chain is a function of the stochastic eigenvalue with the greatest modulus. The eigenvalues of P_α are easy to compute knowing the eigenvalues of the circuit graph (see e.g. [CDS79]).

$$\lambda(P_\alpha)_j = \alpha + (1-\alpha)\cos\frac{2\pi}{n} j \qquad j = 0, ..., n-1 \tag{4}$$

If $\alpha = 1$, $P_\alpha = I$ and thus the diffusion process does not converge. If n is even and $\alpha = 0$, the smallest eigenvalue is equal to -1 and the diffusion process does not converge. For all other values of α, the diffusion process will converge to the uniform load distribution.

Question 3. For which value of α is the convergence of the natural diffusion process the fastest possible?

To answer question 3, we compute the optimal λ_α in the interval $[0,1]$

Theorem 4. *Let* $\lambda_\alpha = \max_{j=1,\ldots,n-1} |\lambda(P_\alpha)_j|$ *and set* $\lambda_{nat} = \min_{\alpha\in[0,1]} \lambda_\alpha$

$$\lambda_{nat} = \begin{cases} \frac{1+\cos\frac{2\pi}{n}}{3-\cos\frac{2\pi}{n}} & n \text{ even} \\ \frac{\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}}{2+\cos\frac{\pi}{n}-\cos\frac{2\pi}{n}} & n \text{ odd} \end{cases} \tag{5}$$

Moreover if α_{nat} *is such that* $\lambda_{nat} = \lambda_{\alpha_{nat}}$*, then*

$$\alpha_{nat} = \begin{cases} \frac{1-\cos\frac{2\pi}{n}}{3-\cos\frac{2\pi}{n}} & n \text{ even} \\ \frac{\cos\frac{\pi}{n}-\cos\frac{2\pi}{n}}{2+\cos\frac{\pi}{n}-\cos\frac{2\pi}{n}} & n \text{ odd} \end{cases} \tag{6}$$

Proof. It is easy to see, that

$$\lambda_\alpha = \begin{cases} \max\{|2\alpha-1|, \alpha+(1-\alpha)\cos\frac{2\pi}{n} & n \text{ even} \\ \max\{|\alpha-(1-\alpha)\cos\frac{\pi}{n}|, \alpha+(1-\alpha)\cos\frac{2\pi}{n}\} & n \text{ odd} \end{cases} \tag{7}$$

Note that

$$|2\alpha-1| = \begin{cases} 1-2\alpha & \text{if } \alpha \leq \frac{1}{2} \\ 2\alpha-1 & \text{if } \alpha \geq \frac{1}{2} \end{cases} \tag{8}$$

If $\alpha \leq \frac{1}{2}$ then it is easy to see that $1-2\alpha \geq \alpha+(1-\alpha)\cos\frac{2\pi}{n}$ if and only if $\alpha \leq \frac{1-\cos\frac{2\pi}{n}}{3-\cos\frac{2\pi}{n}}$ If $\alpha \geq \frac{1}{2}$ then $2\alpha-1 \geq \alpha+(1-\alpha)\cos\frac{2\pi}{n}$ if and only if $\alpha = 1$. For n odd, the proof is similar □

Note 5. If $n = 3$, $\alpha_{\text{nat}} = \frac{1}{3}$ and $\lambda_{\text{opt}} = 0$

3 Diffusion in Cayley Graphs

Definition 6 (Cayley Graph). Let Γ be a finite group and and let S be a symmetric set of generators, i.e. $s \in S \Leftrightarrow s^{-1} \in S$. The Cayley Graph $G(\Gamma, S)$ is the undirected graph with vertex set Γ and where $(g_1, g_2) \in E(G)$ if and only if $g_1^{-1}g_2 \in S$

$G(\Gamma, S)$ is a connected regular graph of order $k = |S|$. Thus at each vertex, the edges may be labeled with the elements of S.

Example 1. Let $\Gamma = (\mathbb{Z}_2)^d$, and set $S = \{(1,0,...,0),(0,1,...,0),...,(0,0,...,1)\}$. Then $G(\Gamma, S) = H_d$ the d-dimensional Cube Graph.

Example 2. Let $\Gamma = \sigma_3$ the symmetric group of the 3 element set $\{1,2,3\}$ and set $S = \{(1,2),(1,2,3),(3,2,1)\}$. Then $G(\Gamma, S)$ is a prism with triangular bottom.

Example 3. Consider $n \geq 2$ be an integer, the dihedral group $\Gamma = < s, t | s^2 = t^2 = (st)^n = 1 >$ of order $2n$ and set $S = \{s, t\}$. Then $G(\Gamma, S)$ is a circuit with $2n$ vertices.

We define in S an equivalence relation by identifying s with s^{-1} ($s \in S$). Let $\hat{S}$ be the set of equivalence classes. We call $\hat{S} = \{\hat{s}_1, \hat{s}_2, ..., \hat{s}_k\}$ the set of **dimensions** of G. Note that if $s \in S$ is an element of order 2 then $\hat{s} = \{s\}$, if not, then $\hat{s} = \{s, s^{-1}\}$. Using $\hat{S}$, we may build a partition $\hat{E} = \{\hat{E}_1, \hat{E}_2, ..., \hat{E}_k\}$ of the edge set $E(G)$ by defining

$$(g_1, g_2) \in \hat{E}_i \text{ if and only if } g_1^{-1}g_2 \in \hat{s}_i \quad i = 1, ..., k \tag{9}$$

If $\hat{s}_i$ corresponds to an element of S of order 2, then $\hat{E}_i$ is a disjoint set of edges. If $\hat{s}_i$ corresponds to an element of S of order $d > 2$, then $\hat{E}_i$ is the union of disjoint circuits (of length d) of G.

Example 4. For the d-dimensional Cube Graph with generator set as in example 1, $\hat{E}_i$ is the set of all edges which are parallel to the i^{th} axis, i.e. $(g_1, g_2) \in \hat{E}_i$ if and only if g_1 and g_2 differ only in the i^{th} component.

Example 5. For the symmetric group of the 3 element set $\{1, 2, 3\}$ with generator set as in example 2, $G(\Gamma, S)$ is a prism with triangular bottom. $\hat{E}_{(1,2)}$ corresponds to the vertical edges of G and $\hat{E}_{(1,2,3)}$ corresponds to the bottom and the top of G.

Example 6. Consider the dihedral group with generator set as in example 3. If the edges of G are numbered from 1 to $2n$, then $\hat{E}_s$ corresponds to the even numbered edges and $\hat{E}_t$ corresponds to the odd numbered edges.

3.1 Load balancing in Cayley graphs

The natural decomposition $\{\hat{E}_1, \hat{E}_2, ..., \hat{E}_k\}$ of a Cayley graph $G(\Gamma, S)$ leads to following load balancing strategy:

Algorithm 3 (Cayley diffusion strategy).

```
WHILE TRUE
  PAR i = 1 FOR processors
    SEQ j = 1 FOR dimensions
      exchange load with direct neighbors in dimension j
```

Instead of distributing the load to all neighbors at the same time, the load is distributed successively in all dimensions. Thus a diffusion step consists in balancing the load (using algorithm 2) along all edges of the elements of the partition $\hat{E}$ successively. A single step of the `WHILE` loop in the Cayley diffusion strategy requires exactly the same number of communications and synchronizations than the corresponding step of the natural diffusion strategy.

Two cases must be considered. If the direction s_j corresponds to a generator s of order 2, then each vertex gives (or receives) the half of the load difference to (from) its single direct neighbor in direction $\hat{s}_j$. If the direction s_j corresponds to a generator s of order $d \geq 2$, then each vertex gives (or receives) $\frac{1-\alpha_0}{2}$ times the load difference to (from) both neighbors in direction $\hat{s}_j$, where α_0 is defined as in theorem 4. Indeed, in this case $\hat{E}_j$ consists in disjoint circuits of length d.

Let $\hat{G}_j$ denote the graph with the same vertex set as G and with edges $\hat{E}_j$, the algorithm may be reformulated as follows:

Algorithm 4 (Cayley diffusion).

```
WHILE TRUE
  PAR i = 1 FOR processors
    SEQ j = 1 FOR dimensions
      exchange load using the fastest natural strategy
      with all direct neighbors in the graph Ĝ_j
```

Let $\hat{A}_j$ denote the adjacency matrix of the graph $\hat{G}_j$. $\hat{A}_j$ is the adjacency matrix of a disjoint union of circuits of length d (if $\hat{s}_j$ corresponds to a generator of order $d > 2$) or the adjacency of a disjoint union of edges (matching) if $\hat{s}_j$ corresponds to a generator of order $d = 2$. The stochastic matrix corresponding to the diffusion in dimension j is the matrix

$$\hat{P}_j = \begin{cases} \alpha I + (1-\alpha)\hat{A}_j & (if) d = 2 \\ \alpha I + \frac{1-\alpha}{2}\hat{A}_j & (if) d > 2 \end{cases} \tag{10}$$

where

$$\alpha = \begin{cases} \frac{1-\cos\frac{2\pi}{d}}{3-\cos\frac{2\pi}{d}} & d \text{ even} \\ \frac{\cos\frac{\pi}{d}-\cos\frac{2\pi}{d}}{2+\cos\frac{\pi}{d}-\cos\frac{2\pi}{d}} & d \text{ odd} \end{cases} \tag{11}$$

The stochastic matrix $\hat{P}$ corresponding to one step of the Cayley diffusion is the product of all partial diffusion matrices in all directions:

$$\hat{P} = \prod_{j=1}^{k} \hat{P}_j \tag{12}$$

$\hat{P}$ is a stochastic matrix, however $\hat{P}$ is normally not symmetric.

Theorem 7. *The Cayley diffusion scheme always converges towards the uniform load distribution.*

Proof. $\hat{P}$ is a product of doubly stochastic matrices, thus $\hat{P}$ is itself doubly stochastic. Since the diagonal elements of the $\hat{P}_j$ are strictly positive, so are the diagonal elements of $\hat{P}$. Since G is connected, it follows that $\hat{P}$ is primitive, i.e.$\hat{P}$ has only one eigenvalue of modulus 1. □

Note 8. The eigenvalues of $\hat{P}$ are not independent of the numbering of the subgraphs $\hat{G}_j$. However, it is well known that given any two square matrices A and B, AB and BA have the same eigenvalues. Thus cyclic permutations of the product $\hat{P}$ will have the same eigenvalues.

3.2 Examples

In this section, we compare both diffusion schemes for some particular Cayley graphs, e.g. the d-dimensional cube, the circuit with an even number of vertices and some Cayley graphs from the symmetric group. We show that the Cayley diffusion is faster than the natural diffusion in all cases.

Example 7. Let G be square graph, i.e. the Cayley graph of $(\mathbb{Z}_2)^2$ with generators $s_1 = (1, 0)$ and $s_2 = (0, 1)$, then

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{13}$$

$$\hat{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{14}$$

$$\hat{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{15}$$

This results in following stochastic matrices

$$P = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{16}$$

$$\hat{P}_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{17}$$

$$\hat{P}_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{18}$$

$$\hat{P} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{19}$$

It follows that the Cayley diffusion process converges after a single step whereas the stochastic eigenvalue of greatest modulus of the natural diffusion process is $\lambda = \frac{1}{3}$ resulting in slow convergence.

Example 8. The result of example 7 extends to the Cube graph of dimension d (see example 1). The stochastic eigenvalue of greatest modulus of the best possible natural diffusion process is $\lambda = 1 - \frac{2}{d+1}$ [Cyb89]. The Cayley diffusion process converges after a single step. We give here an outline of the proof. The Cayley diffusion consists of d diffusion steps (successively along the edges of all dimensions) it follows that after j steps, all vertex pairs incident to edges in dimension $1, ..., j$ have the same load. In the next step, the load of both ends of these pairs may change, but by the same amount. The proof follows by induction.

Example 9. For the prism of example 2, the Cayley diffusion converges in one single step. Indeed, after the diffusion step in the direction $(1, \hat{2}, 3)$ all the vertices of the top and all vertices of the bottom have the same load. The diffusion step in the direction of $(1\hat{,}2)$ equalizes the load of the top and of the bottom. It is easy to verify that the natural diffusion scheme does not converge in finite time.

Example 10. Let $\Gamma = \sigma_4$ the symmetric group of the 4 element set $\{1,2,3,4\}$ and set $S = \{(2,4),(1,2,3),(1,3,2)\}$. Then $G(\Gamma, S)$ is a cube with truncated vertices. The stochastic eigenvalue with greatest modulus of the Cayley diffusion has value $\frac{2}{3}$ whereas that of the natural diffusion process has value 0.854[1].

Example 11. Let $\Gamma = \sigma_4$ the symmetric group of the 4 element set $\{1,2,3,4\}$ and set $S = \{(1,2,3),(1,3,2),(1,2,4,3),(1,3,4,2)\}$. Then $G(\Gamma, S)$ is a cube with truncated edges. The stochastic eigenvalue with greatest modulus of the Cayley diffusion has value 0.556 whereas that of the natural diffusion process has value 0.75[1].

Example 12. The circuit graph with $2n$ $(n \geq 3)$ vertices may be considered as the Cayley graph of the dihedral group with generator set $S = \{s, t\}$ (see example 3). The stochastic eigenvalue of greatest modulus of the Cayley diffusion process is $\lambda = \frac{1}{2}(1 + \cos\frac{2\pi}{n})$ (see note 11). Its is easy to verify that λ is smaller than the corresponding eigenvalue of the natural diffusion. We will see in the next section that faster diffusion is possible in circuits with an even number of vertices.

4 Optimal load balancing in circuits with an even number of vertices

The Cayley diffusion strategy leads to fast convergence provided the order of the generators of the graph is small. Indeed, if s is a generator of order d, $\hat{A}_{\hat{s}}$ is the adjacency matrix of a disjoint union of circuits of length d. Unfortunately, the convergence of the natural diffusion in the circuits may be very slow if d is big.
In this section, we show that it is possible to achieve faster convergence, at least in circuits with an even number of vertices. The method proposed corresponds to a subsequent labelling (coloring) of the edges in the circuits with an even number of vertices.

4.1 Optimal diffusion scheme in circuits with an even number of vertices

Let C_{2n} be a circuit graph with $2n$ $(n \geq 2)$ vertices, considered as the Cayley graph of the dihedral group with generator set $S = \{s, t\}$ (see example 3). In the examples of section 3.2, we have seen that the Cayley diffusion strategy may achieve faster convergence than the natural diffusion scheme. However it is not the fastest strategy. Let $\hat{A}_s$ and $\hat{A}_t$ be the the adjacency matrices corresponding to the subgraphs in the both dimensions. For reason of simplicity, we choose following numbering of the vertices of the circuit: $1, 3, ..., 2n-1, 2, 4, ..., 2n$.

[1] This example was computed using MAPLE

$$\hat{A}_s = \begin{pmatrix} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{pmatrix} \tag{20}$$

$$\hat{A}_t = \begin{pmatrix} & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & & & & \end{pmatrix} \tag{21}$$

Both $\hat{A}_s$ and $\hat{A}_t$ are involutions, i.e. $\hat{A}_s^2 = \hat{A}_t^2 = I$.
Let $\boldsymbol{C}_n$ be the adjacency matrix of the directed circuit with n vertices and let I_n denote the identity matrix. Using the Cayley diffusion strategy, each edge in $\hat{A}_s$ and $\hat{A}_t$ would get the weight $\frac{1}{2}$ since s and t are generators of order 2. We solve here the more complex problem of computing the optimal diffusion speed when the edges of $\hat{A}_s$ have weight $1-\alpha$, and those of $\hat{A}_t$ weight $1-\beta$.
Let let $\hat{P}_s(\alpha)$ resp. $\hat{P}_t(\beta)$ be the associated diffusion matrices. It follows that

$$\hat{P}_s(\alpha) = \begin{pmatrix} \alpha I_n & (1-\alpha)I_n \\ (1-\alpha)I_n & \alpha I_n \end{pmatrix} \tag{22}$$

and

$$\hat{P}_t(\beta) = \begin{pmatrix} \beta I_n & (1-\beta)\boldsymbol{C}_n^{-1} \\ (1-\beta)\boldsymbol{C}_n & \beta I_n \end{pmatrix} \tag{23}$$

Let $\hat{P}(\alpha,\beta) = \hat{P}_s(\alpha)\hat{P}_t(\beta)$. Then

$$\hat{P}(\alpha,\beta) = \begin{pmatrix} \alpha\beta I_n + (1-\alpha)(1-\beta)\boldsymbol{C}_n & \alpha(1-\beta)\boldsymbol{C}_n^{-1} + \beta(1-\alpha)I_n \\ \alpha(1-\beta)\boldsymbol{C}_n + \beta(1-\alpha)I_n & \alpha\beta I_n + (1-\alpha)(1-\beta)\boldsymbol{C}_n^{-1} \end{pmatrix} \tag{24}$$

Theorem 9. *Let $c \in [-1,1]$, $h_c = \alpha\beta + (1-\alpha)(1-\beta)c$ and $g = (1-2\alpha)(1-2\beta)$. The eigenvalues of $\hat{P}(\alpha,\beta)$ are given by*

$$\lambda_j^{\mp} = h_c \mp \sqrt{h_c^2 - g} \tag{25}$$

where $c = \cos\frac{2\pi j}{n}$, $j = 1, \ldots, n$

Proof. We have to solve the characteristic equation

$$|P(\alpha,\beta) - \lambda I_{2n}| = 0 \tag{26}$$

Before solving the characteristic equation, we recall following well known result

Lemma 10. *Let A, B, C and D be commuting $n \times n$ matrices. Then*

$$\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \lambda I_{2n} \right| = 0 \tag{27}$$

if and only if

$$|(A - \lambda I_n)(D - \lambda I_n) - CB| = 0 \tag{28}$$

□

Since all blocks of $P(\alpha, \beta)$ commute, lemma 10 applies and we get following equation

$$\left|(1-2\alpha)(1-2\beta)I_n + \lambda((\lambda - 2\alpha\beta)I_n - (1-\alpha)(1-\beta)(\boldsymbol{C}_n + \boldsymbol{C}_n^{-1}))\right| = 0 \tag{29}$$

Let $C_n = \boldsymbol{C}_n + \boldsymbol{C}_n^{-1}$ and assume that $\lambda \neq 0$ i.e. $\alpha \neq \frac{1}{2} \wedge \beta \neq \frac{1}{2}$. Then

$$|P(\alpha, \beta) - \lambda I_{2n}| = 0 \tag{30}$$

if and only if

$$\left|(\lambda - 2\alpha\beta + \frac{1}{\lambda}(1-2\alpha)(1-2\beta))I_n - (1-\alpha)(1-\beta)C_n\right| = 0 \tag{31}$$

Let

$$\mu = \lambda - 2\alpha\beta + \frac{1}{\lambda}(1-2\alpha)(1-2\beta) \tag{32}$$

it follows that

$$|P(\alpha, \beta) - \lambda I_{2n}| = 0 \Longleftrightarrow |(1-\alpha)(1-\beta)C_n - \mu I_n| = 0 \tag{33}$$

Since the eigenvalues of the circuit graph C_n are $2\cos\frac{2\pi j}{n} \quad i = j, .., n$ it follows that

$$\mu = 2(1-\alpha)(1-\beta)\cos\frac{2\pi j}{n} \quad j = 1, ..., n \tag{34}$$

i.e.

$$\lambda - 2\alpha\beta + \frac{1}{\lambda}(1-2\alpha)(1-2\beta) = 2(1-\alpha)(1-\beta)\cos\frac{2\pi j}{n} \quad j = 1, .., n \tag{35}$$

hence

$$\lambda_j^{\mp} = h_c \mp \sqrt{h_c^2 - g} \tag{36}$$

where $c = \cos\frac{2\pi j}{n}$, $j = 1, ..., n$ □

Note 11. If $\alpha = \frac{1}{2}$ then it is easy to see that

$$\lambda_j^+ = \beta + (1-\beta)\cos\frac{2\pi j}{n} \qquad j = 1, ..., n \tag{37}$$

$$\lambda_j^- = 0 \qquad j = 1, ..., n \tag{38}$$

i.e. the non zero eigenvalues are the eigenvalues of the natural diffusion in a circuit with n vertices. Moreover, this last result is compatible with equation 36.

Corollary 12. *This diffusion scheme in C_{2n} is at least as fast as the natural diffusion in C_n* □

Theorem 13. *If $n = 2$ and $\alpha = \beta = \frac{1}{2}$ it follows from note 11 that all stochastic eigenvalues are zero, thus the diffusion converges towards the uniform distribution in a finite number of steps.*
If $n = 3$ and $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ or $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$ it follows from note 11 that all stochastic eigenvalues are zero, thus the diffusion converges towards the uniform distribution in a finite number of steps. □

4.2 Main Results

Let λ be the function from the unit square onto $[-1, 1]$ defined by

$$\lambda : (\alpha, \beta) \longrightarrow \max_{j=1,..,n-1} \{|\lambda_j^+|, |\lambda_j^-|\} \tag{39}$$

Theorem 14. *Let $n > 3$. The optimal diffusion is given by the pairs $(\alpha_{opt}, \beta_{opt}) \in [0,1]^2$ such that*

$$(\alpha_{opt}, \beta_{opt}) \in \lambda^{-1}\left(\min_{(\alpha,\beta)\in[0,1]^2} \lambda(\alpha, \beta)\right) \tag{40}$$

Proof. Since $\lambda_n^- = (1-2\alpha)(1-2\beta) = g$, the proof follows directly from lemma 20 □

Theorem 15. *Let $n > 3$ and set $n = 2m$ if n is even and $n = 2m+1$ if n is odd. The optimal diffusion is given by the pairs $(\alpha_{opt}, \beta_{opt}) \in [0,1]^2$ such that*

$$(\alpha_{opt}, \beta_{opt}) \in \lambda^{-1}\left(\min_{(\alpha,\beta)\in[0,1]^2} \max\{|\lambda_1^+|, |\lambda_m^-|\}\right) \tag{41}$$

Proof. Follows from $\cos\frac{2\pi}{n} \geq \cos\frac{2\pi j}{n} \geq \cos\frac{2\pi m}{n} \quad (j = 1, ..., n-1)$ and lemmas 21 and 22 □

Theorem 16. *Let $n = 2m+1$, $n > 3$ be an odd number. The optimal diffusion is given by the two pairs $(\alpha_{opt}, \beta_{opt})$ and $(\beta_{opt}, \alpha_{opt})$ such that*

$$\alpha_{opt} = \frac{\left(\sqrt{1+c} + \sqrt{1+d}\right)\left(\sqrt{1-c^2} - \sqrt{1-d^2}\right)}{\left(\sqrt{1-c} + \sqrt{1-d}\right)(c-d)} \tag{42}$$

$$\beta_{opt} = \frac{\left(\sqrt{1+c} - \sqrt{1+d}\right)\left(\sqrt{1-c^2} - \sqrt{1-d^2}\right)}{\left(\sqrt{1-c} + \sqrt{1-d}\right)(c-d)} \tag{43}$$

where $c = \cos\frac{2\pi}{n}$ *and* $d = \cos\frac{2\pi m}{n} = -\cos\frac{\pi}{n}$. *Moreover*

$$\lambda_{opt} = \frac{1 - 2\sin\left(\frac{\pi}{2n}\right)}{1 + 2\sin\left(\frac{\pi}{2n}\right)} \tag{44}$$

Theorem 17. *Let* $n = 2m$, $n > 2$ *be an even number. The optimal diffusion is given by the pair* $(\alpha_{opt}, \alpha_{opt})$ *such that*

$$\alpha_{opt} = \frac{\sin\left(\frac{\pi}{n}\right)}{1 - \sin\left(\frac{\pi}{n}\right)} \tag{45}$$

Moreover

$$\lambda_{opt} = \frac{1 - \sin\left(\frac{\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)} \tag{46}$$

Proof. **(Overview)** Let first consider the equation

$$h_1(\alpha,\beta) = -h_m(\alpha,\beta) \tag{47}$$

The solutions may be considered as a function

$$\phi : \beta \longrightarrow \frac{(1-\beta)(c+d)}{(1-\beta)(c+d) - 2\beta} \tag{48}$$

It is easy to see that

$$\lambda(\alpha,\beta) = \begin{cases} \lambda_m^-(\alpha,\beta) & \text{if } \alpha \le \frac{(1-\beta)(c+d)}{(1-\beta)(c+d)-2\beta} \\ \lambda_1^+(\alpha,\beta) & \text{else} \end{cases} \tag{49}$$

Moreover consider the equation

$$\Delta_1(\alpha,\beta) = 0 \tag{50}$$

The solution in $[0, \frac{1}{2}]^2$ may also be considered as a function

$$\psi : \beta \longrightarrow \frac{\left(c(1-\beta) - \beta\right)^2 + (1-\beta)\left(c\,\beta - (1-\beta) + \sqrt{1-2\,\beta}\sqrt{1-c^2}\right)}{\left(c(1-\beta)-\beta\right)^2} \tag{51}$$

If n is even then for any (fixed) $\beta \in [0,1]$

$$\lambda_1^+(\phi(\beta),\beta) \le \lambda(\alpha,\beta) \qquad (\forall \alpha \in [0,1]) \tag{52}$$

i.e. the optimal solution lies on the graph of ϕ, moreover $\lambda_1^+(\beta,\phi(\beta)) \in I\!R \quad (\forall \beta \in [0,1])$

Figure 1 shows the graphs of ϕ (`h(a,b,c) = -h(a,b,d)`) and ψ (`disc(a,b,c) = 0`). Note that both graphs intersect in exactly one point of the unit square.

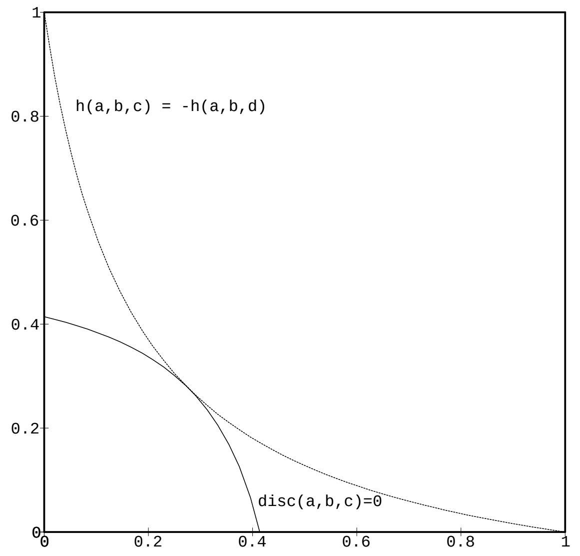

Fig. 1. n even

If n is odd then for any(fixed) $\beta \in [0,1]$

$$\lambda_1^+(\phi(\beta),\beta) \leq \lambda(\alpha,\beta) \qquad (\forall\alpha \in [0,1]) \tag{53}$$

provided $\lambda_1^+(\phi(\beta),\beta) \in I\!R$ and

$$\sqrt{g(\psi(\beta),\beta)} \leq \lambda(\alpha,\beta) \qquad (\forall\alpha \in [0,1]) \tag{54}$$

provided $\lambda_1^+(\phi(\beta),\beta) \in \mathbb{C} \setminus I\!R$.
Figure 2 shows the graphs of ϕ (`h(a,b,c) = -h(a,b,d)`), ψ (`disc(a,b,c) = 0`) and of the solution of the equation $\Delta_d = 0$ (`disc(a,b,d) = 0`). Note that the three graphs intersect in exactly two points of the unit square.
The function $h_c(\alpha,\phi(\alpha))$, $c = \cos\frac{2\pi}{n}$ takes its minimum at $\alpha_0 = \phi(\alpha_0)$, i.e. on the line $\alpha = \beta$. Moreover

$$\frac{\partial h_c(\alpha,\phi(\alpha))}{\partial\alpha} \leq 0 \qquad \text{if } \alpha \in [0,\alpha_0] \tag{55}$$

and

$$\frac{\partial h_c(\alpha,\phi(\alpha))}{\partial\alpha} \geq 0 \qquad \text{if } \alpha \in [\alpha_0,\phi(0)] \tag{56}$$

Since $\lambda_1^+(\alpha,\phi(\alpha)) \in I\!R$ if n is even, it takes its optimal value at α_0.

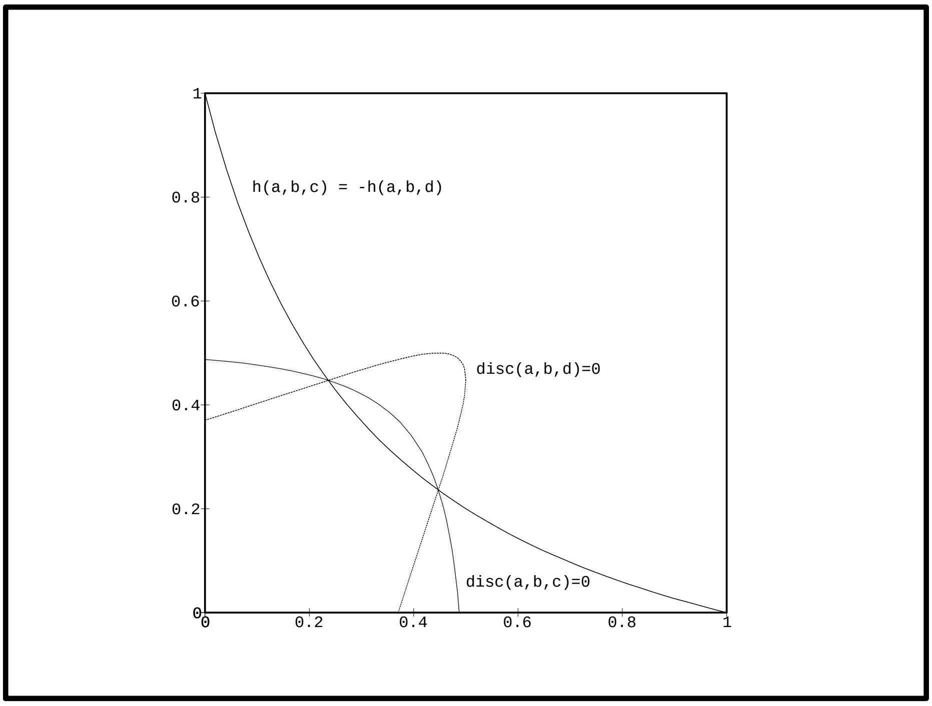

Fig. 2. n odd

If n is odd, $\lambda_1^+(\alpha, \phi(\alpha)) \in \mathbb{C} \setminus I\!R$ near the line $\alpha = \beta$. The real valued optima are thus taken at the intersection (α_0, β_0) and (β_0, α_0) of the graphs of ϕ and ψ. Assume that $\alpha_0 \leq \beta_0$.
In the complex domain λ_1^+ takes its minimal modulus on the graph of ψ because $\frac{\partial g}{\partial \alpha} \leq 0$ (β fixed) in the domain $[0, \frac{1}{2}]^2$.
In the interval $[\alpha_0, \beta_0]$, $\lambda_1^+(\alpha, \psi(\alpha))$ takes its minima at α_0 and β_0.
See appendix A for detailed proofs. □

It is surprising that the edges labeled with $\hat{s}$ do not have the same weight than those labeled with $\hat{t}$. We propose following explanation: The first weight is responsible for fast local balancing, the second for fast diffusion in the whole circuit. e.g. in C_6, the first weight is $\frac{1}{2}$ and leads to optimal local load balance, the second weight is $\frac{2}{3}$ and leads to fast diffusion in the rest of the circuit. Both goals are obviously contradictory.

Theorem 18. *Asymptotically, the Cayley diffusion in $C_{(2n)^2}$ achieves the same convergence speed than the natural diffusion in C_{2n}.*

Proof. Let $\lambda_{\text{nat}}(n)$ be the natural diffusion speed in C_{2n} and $\lambda_{\text{opt}}(m)$ be the optimal diffusion speed in C_{2m}. It is easy to see that

$$\lambda_{\text{nat}}(n) = \frac{1 + \cos\frac{\pi}{n}}{3 - \cos\frac{\pi}{n}} = \frac{1 - \sin^2\frac{\pi}{2n}}{1 + \sin^2\frac{\pi}{2n}} \tag{57}$$

Following inequalities are equivalent

$$\lambda_{\text{nat}}(n) \geq \lambda_{\text{cal}}(m) \tag{58}$$

$$\frac{1-\sin^2\frac{\pi}{2n}}{1+\sin^2\frac{\pi}{2n}} \geq \frac{1-\sin\frac{\pi}{2m}}{1+\sin\frac{\pi}{2m}} \tag{59}$$

$$\sin\frac{\pi}{2m} \geq \sin^2\frac{\pi}{2n} \tag{60}$$

Since $x - \frac{x^3}{3!} \leq \sin x \leq x$ $(0 \leq x \text{ small})$, $\sin x \geq \sin^2 Y$ follows from $x \geq y^2$.
Hence

$$2m \leq O\left(\frac{(2n)^2}{\pi}\right) \tag{61}$$

□

5 Conclusion

Cayley graphs make good choices for multiprocessor networks, they are vertex symmetric, and most standard networks can be formulated in terms of Cayley graphs. Moreover, we have proposed a new load diffusion scheme for Cayley graphs and have seen that its convergence is better than the natural scheme, at least for some particular graphs. However, this strategy is not always the best one. For example, the natural diffusion in the complete graph is always better or equal to the Cayley diffusion strategy[2]. The reason is probably that the generator set of the Cayley graph is not minimal:

Conjecture 19. *The convergence of the Cayley diffusion scheme in the Cayley graph $G(\Gamma, S)$ is faster than that of the natural diffusion scheme provided S is a minimal symmetric set of generators.*

We have done a lot of numerical computations with Cayley graphs from the symmetric groups. The Cayley diffusion scheme was still better than the natural one.
Using the results of section 4 it is possible to improve the Cayley diffusion strategy. Indeed if the edges belonging to generators of even order $d > 2$ are subsequently labeled like in theorems 16 and 17, the convergence speed may be faster. Unfortunately, there is no canonical way to label such edges.
Another interesting area consists in sums[3] of graphs [CDS79]. We can prove that if there is a fast diffusion scheme for each component of the sum, the there exist a fast diffusion scheme in the sum. To achieve this goal it is sufficient to use a strategy similar to the Cayley diffusion strategy (the dimensions correspond here to the components of the graph sum). An interesting result is that in any sum of

[2] If Γ is a group of order n with unit e then $G = (\Gamma, \Gamma \setminus \{e\})$ is a complete graph

[3] If $G_1 = (E_1, V_1)$ and $G_2 = (E_2, V_2)$ are graphs, the sum $G_1 + G_2$ is the graph with vertex set $V_1 \times V_2$. The vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if either $x_1 = y_1$ and $(x_2, y_2) \in E_2$ or $x_2 = y_2$ and $(x_1, y_1) \in E_1$. The d dimensional cube graph is the sum of d paths with 2 vertices.

hexagonal graphs, square graphs and paths with 2 vertices, uniform load distribution is reached after a finite number of local diffusion steps. This will be discussed in detail in a subsequent paper.
A further interesting area to study would be the graphs for which there exist a partition of its edge set into perfect matchings. This would lead to similar load balancing schemes. Indeed, during the redaction of this article, the author became aware of an article of Xu and Lau [XL92]. In this article, a similar load balancing method for colored graphs is discussed extensively.

A Utilities

This appendix contains all necessary stuff to prove theorems 16 and 16. The most computations have been made with MAPLE and subsequently verified by the author.

Let $x \in [-1, 1]$. We define following functions from the unit square $[0, 1]^2$ onto $[-1, 1]$:

$$h_x = \alpha\beta + (1-\alpha)(1-\beta)x \tag{62}$$

$$g = (1-2\alpha)(1-2\beta) \tag{63}$$

$$\Delta_x = h_x^2 - g \tag{64}$$

A.1 Moduli

Lemma 20. *If $x > 0$ then*

$$|h_x + \sqrt{h_x^2 - g}| \geq \sqrt{|g|} \geq |g| \tag{65}$$

Proof. If $\Delta_x < 0$ then $h_x + \sqrt{h_x^2 - g} \in \mathbb{C} \setminus I\!R$ and $|h_x + \sqrt{h_x^2 - g}| = \sqrt{|g|} \geq |g|$ because $(\alpha, \beta) \in [0, 1]^2$.
If $\Delta_x \geq 0$ then either $|g| = g$ and the proof follows from $\sqrt{|g|} \leq h_x + \sqrt{h_x^2 - g} \Leftrightarrow 0 \leq (h_x - \sqrt{|g|}) + \sqrt{h_x^2 - g}$ (both terms are positive) or $|g| = -g$ and the proof follows from $\sqrt{|g|}^2 = -g \leq (h_x + \sqrt{h_x^2 - g})^2 \Leftrightarrow 0 \leq 2h_x^2 + 2h_x\sqrt{h_x^2 - g}$ □

Lemma 21. *Let $c \in [0, 1]$, $|d| \leq c$ then*

$$|h_d \mp \sqrt{\Delta_d}| \leq |h_c + \sqrt{\Delta_c}| \tag{66}$$

Proof. It is easy to see that $-h_c \leq h_d \leq h_c$. It follows that $\Delta_d \leq \Delta_c$.
If $\Delta_d < 0$ then $h_d \mp \sqrt{\Delta_d} \in \mathbb{C} \setminus I\!R$ and the proof follows from lemma 20.
Let $\Delta_d \geq 0$. The proof follows from $h_c + \sqrt{\Delta_c} \geq h_d + \sqrt{\Delta_d} \geq h_d - \sqrt{\Delta d} \geq -h_c - \sqrt{\Delta_c}$ □

Lemma 22. *Let $c \in [0, 1]$, $d \in [-1, 0]$. Then either*

$$|h_d \mp \sqrt{\Delta_d}| \leq |h_c + \sqrt{\Delta_c}| \tag{67}$$

if $h_c \geq h_d \geq -h_c$ or

$$|h_c \mp \sqrt{\Delta_c}| \leq |h_d - \sqrt{\Delta_d}| \tag{68}$$

if $h_d \leq -h_c$

Proof. Follows directly from lemma 21 □

A.2 Values

Lemma 23. *Let $x \in [-1, 1]$ then if $\alpha \geq \frac{1}{2}$ or $\beta \geq \frac{1}{2}$ then $h_x \in IR$.*

Proof. If $\alpha \geq \frac{1}{2}$ and $\beta \leq \frac{1}{2}$ or $\alpha \leq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$ then $g \leq 0$, hence $\Delta_x \geq 0$.
Let $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$. Since $-1 \leq x \leq 1$, $h_x \geq \alpha\beta - (1-\alpha)(1-\beta) = \alpha + \beta - 1 \geq 0$.
It follows that $\Delta_x \geq (\alpha+\beta-1)^2 - (1-2\alpha)(1-2\beta) = (\alpha-\beta)^2 \geq 0$. □

Corollary 24. *Let $x \in [-1, 1]$. The function h_x may only take complex values if $\alpha \leq \frac{1}{2}$ and $\beta \leq \frac{1}{2}$* □

Let $0 \leq c < 1$, $g_0 \in [0, 1]$, $\alpha, \beta \in [0, \frac{1}{2}]^2$ and consider the solutions of $g_{\alpha,\beta} = g_0$ as a function δ of α, i.e.

$$\delta : \alpha \longrightarrow \frac{1}{2}\left(1 - \frac{g_0}{1-2\alpha}\right) \tag{69}$$

Lemma 25. *Suppose that $\Delta_c(\alpha_0, \delta(\alpha_0)) = 0$ and let $\beta_0 = \delta(\alpha_0)$. We may assume that $\alpha_0 \leq \beta_0$. Then Δ_c is positive between its zeros α_0 and β_0.*

Proof. $\Delta_c(\alpha, \delta(\alpha)) = h_c^2(\alpha, \delta(\alpha)) - g(\alpha, \delta(\alpha)) = h_c^2(\alpha, \delta(\alpha)) - g_0$. Thus the inequality $\Delta_c(\alpha_0, \delta(\alpha_0)) \geq 0$ is equivalent to $h_c^2(\alpha, \delta(\alpha)) \geq g_0$ Since h_c is positive we have to solve $h_c(\alpha, \delta(\alpha)) \geq \sqrt{g_0}$ It is easy to see that following inequalities are equivalent

$$h_c(\alpha, \delta(\alpha)) \geq \sqrt{g_0} \tag{70}$$

$$\frac{1}{2}\frac{2\alpha^2(c-1) + \alpha(1-g_0-c(3+g_0)) + c(1+g)}{1-2\alpha} \geq \sqrt{g_0} \tag{71}$$

$$2\alpha^2(c-1) + \alpha(1-g_0-c(3+g_0)+4\sqrt{g_0}) + c(1+g) - 2\sqrt{g_0} \geq 0 \tag{72}$$

We have used the fact that $0 \leq \alpha \leq \frac{1}{2}$.
Since $1 - c > 0$ the inequality cannot be identically 0, moreover, the inequality is verified between the roots of the last quadratic function □

Corollary 26. *Let $\alpha_0 \leq \beta_0$ be two solutions of $\Delta_c(\alpha_0, \beta_0) = 0$ and set $g_0 = h_c^2(\alpha_0, \beta_0)$. On the graph of $\Delta_c = 0$, g takes its minimum at α_0 and β_0 in the interval $[\alpha_0, \beta_0]$.*

Proof. Let $\alpha \in [\alpha_0, \beta_0]$. Since $\Delta_c(\alpha, \delta(\alpha)) > 0$, the value of g on the graph of $\Delta_c = 0$ at α must be greater than g_0 because $\frac{\partial g}{\partial \alpha} \leq 0$ and $\frac{\partial g}{\partial \beta} \leq 0$ in $[0, \frac{1}{2}]^2$ □

A.3 Derivatives

Let $c \in [0, 1]$. We may consider h_c and g as functions of α, i.e. as $h_c : \alpha \longrightarrow (\beta - (1-\beta)c)\alpha + (1-\beta)c$ and $g : \alpha \longrightarrow 2(2\beta-1)\alpha + (1-2\beta)$ defined in the unit interval $[0, 1]$ ($\beta \in [0, 1]$ fixed).

Lemma 27. *Let $f_c : \alpha \longrightarrow h_c(\alpha) + \sqrt{h_c^2(\alpha) - g(\alpha)}$. Then if $f_c(\alpha) \in IR$ then f_c is positive and $\frac{\partial f_c}{\partial \alpha} \geq 0$.*

Proof. Let $f_c(\alpha) \in IR$. It is easy to see that

$$\frac{\partial f_c}{\partial \alpha} = \frac{(\beta - (1-\beta)c)f_c + 1 - 2\beta}{\sqrt{h_c^2 - g}} \tag{73}$$

We consider following three cases

1. If $0 \leq \beta \leq \frac{c}{c+1}$, i.e. if $\beta - (1-\beta)c \leq 0$ then since $0 \leq f_c \leq 1$ it follows that $(\beta - (1-\beta)c)f_c \geq \beta - (1-\beta)c$. Since $\beta - (1-\beta)c + 1 - 2\beta = (1-\beta)(1-c) \geq 0$ the proof follows.
2. Let $\frac{c}{c+1} \leq \beta \leq \frac{1}{2}$. The partial derivative of f_c may be written as
$$\frac{\partial f_c}{\partial \alpha} = \beta - (1-\beta)c + \frac{(\beta - (1-\beta)c)h_c + 1 - 2\beta}{\sqrt{h_c^2 - g}} \tag{74}$$
Since $\beta - (1-\beta)c \geq 0$, $h_c \geq 0$ and $1 - 2\beta \geq 0$ the proof follows.
3. Let $\frac{1}{2} \leq \beta \leq 1$ set $\gamma = \beta - (1-\beta)c$. We show that
$$\left(\frac{\gamma h_c + 1 - 2\beta}{\sqrt{h_c^2 - g}}\right)^2 \leq \gamma^2 \tag{75}$$
This is equivalent to
$$\frac{\gamma^2 h_c^2 + 2(1-2\beta)h_c + (1-2\beta)^2}{h_c^2 - g} \leq \gamma^2 \tag{76}$$
$$\gamma^2 h_c^2 + 2(1-2\beta)h_c + (1-2\beta)^2 \leq \gamma^2 h_c^2 - \gamma^2 g \tag{77}$$
$$(1-2\beta)\left(2\gamma h_c + (1-2\beta)\right) \leq -\gamma^2(1-2\alpha)(1-2\beta) \tag{78}$$
$$2\gamma h_c + (1-2\beta) \geq \gamma^2(1-2\alpha) \tag{79}$$
$$2\gamma(\gamma\alpha + (1-\beta)c) + 1 - 2\beta \geq 2\gamma^2\alpha - \gamma^2 \tag{80}$$
$$2\gamma(1-\beta)c + 1 - 2\beta \geq -\gamma^2 \tag{81}$$
$$(1-\beta)^2(1-c^2) \geq 0 \tag{82}$$
We have used the facts that $h_c^2 - g \geq 0$ (f_c real valued) and $(1-2\beta) \leq 0$

□

Let $d \in [-1, 1]$. We may consider h_d as a function of β, i.e. as $h_d : \alpha \longrightarrow (\beta - (1-\beta)d)\alpha + (1-\beta)d$ defined in the unit interval $[0, 1]$.

Lemma 28. *Let $f_d : \alpha \longrightarrow h_d(\alpha) - \sqrt{h_d^2(\alpha) - g(\alpha)}$. Then if $h_d \leq 0$ and $f_d(\alpha) \in IR$ then $\frac{\partial f_d}{\partial \alpha} \geq 0$.*

Proof. The proof is analogue to that of lemma 27. Note that the proof reduces to 2 cases only because $\frac{d}{1+d} \leq 0$. □

Let $c \in [0, 1]$, $d \in [-1, 1]$ and assume that $c \leq |d|$. We define the function $\phi : [0, 1] \longrightarrow IR$ by

$$\phi(\alpha) = \frac{(1-\alpha)(c+d)}{(1-\alpha)(c+d) - 2\alpha} \tag{83}$$

Lemma 29. *Let* $\tilde{h}_c(\alpha, \phi(\alpha))$, *then*

$$\frac{\partial \tilde{h}_c}{\partial \alpha} \leq 0 \qquad \textit{if } \alpha \in [0, \frac{c+d+\sqrt{-2(c+d)}}{2+c+d}] \tag{84}$$

$$\frac{\partial \tilde{h}_c}{\partial \alpha}(\frac{c+d+\sqrt{-2(c+d)}}{2+c+d}) = 0 \tag{85}$$

$$\frac{\partial \tilde{h}_c}{\partial \alpha} \geq 0 \qquad \textit{if } \alpha \in \left[\frac{c+d+\sqrt{-2(c+d)}}{2+c+d}\right] \tag{86}$$

Proof. It is easy to see that

$$\frac{\partial \tilde{h}_c}{\partial \alpha} = \frac{(1-\alpha)^2(c+d)+2\alpha^2}{((1-\alpha)(c+d)-2\alpha)^2} \tag{87}$$

Thus the sign of the derivative $\frac{\partial \tilde{h}_c}{\partial \alpha}$ is the sign of the numerator $(1-\alpha)^2(c+d)+2\alpha^2 = (2+c+d)\alpha^2 - 2(c+d)\alpha + c + d$.
In $[0,1]$ the numerator is zero at $\frac{c+d+\sqrt{-2(c+d)}}{2+c+d}$ $(c+d \leq 0)$. And the proof follows from $2+c+d > 0$, $\frac{c+d-\sqrt{-2(c+d)}}{2+c+d} < 0$ and $\frac{c+d+\sqrt{-2(c+d)}}{2+c+d} > 0$ □

References

[BBK91] J.E. Boillat, F. Brugé, and P.G. Kropf. A Dynamic Load Balancing Algorithm for Molecular Dynamics Simulation on Multi–Processor Systems. *Journal of Computational Physics*, 96:1 – 14, 1991.

[Boi90] J.E. Boillat. Load Balancing and Poisson Equation in a Graph. *Concurrency: Practice and Experience*, 2(4), 1990.

[CDS79] D.M. Cvetkovič, M. Doob, and H. Sachs. *Spectra of graphs.* Academic Press, New York, 1979.

[Cyb89] G. Cybenko. Dynamic Load Balancing for Distributed Memory Multiprocessors. *J. of Parallel and Distributed Computing*, 7:279–301, 1989.

[ELZ86a] D.E. Eager, E.D. Lazowska, and J. Zahorjan. A Comparison of Receiver-Initiated and Sender-Initiated Adaptive Load Sharing. *Performance Evaluation*, 6:53–68, 1986.

[ELZ86b] D.E. Eager, E.D. Lazowska, and J. Zahorjan. Adaptive Load Sharing in Homogenous Distributed Systems. *IEEE Transactions on Software Engeneering*, SE–12(5):662–675, 1986.

[Hać89] A. Hać. Load balancing in distributed systems: A summary. *Performance Evaluation Review*, 16(2-4):17–19, February 1989.

[MTS90] R. Mirchandaney, D. Towsley, and J.A. Stankovic. Adaptive Load Sharing in Heterogenous Distributed Systems. *Journal of Parallel and Distributed Computing*, 9:331–346, 1990.

[XL92] Z. Xu, C and C.M. Lau. Analysis of the Generalized Dimension Exchange Method for Dynamic Load Balancing. *Journal of Parallel and Distributed Computing*, 16:385–393, 1992.

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