Analytical Cuts in Propositional Linear-Time Temporal Logic

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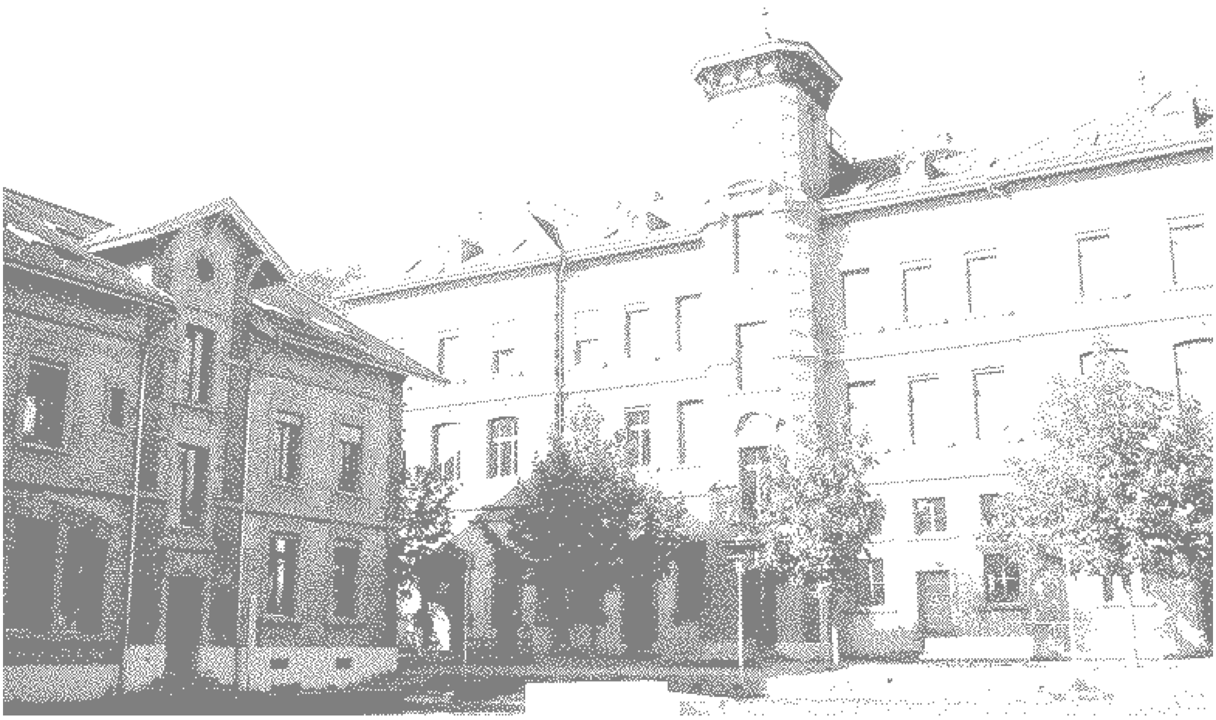

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Abstract

There is no completely satisfying cut-free axiomatization for Propositional Linear Time Temporal Logic, henceforth PLTL [1, 2]. In this paper we aim at a more modest result: we show that it is possible to limit the cuts to a subset of the closure of the sequent whose validity is to be proven. The resulting set is similar to *Fischer-Ladner* cuts [3].

Contents

1 Introduction

There are no completely satisfying cut-free axiomatizations of PLTL. Some approaches use ω-rules [4], others incorporate rules that are as problematic as cut [5]. An interesting approach is that of annotated sequent systems [6], but it entails other problems: even weakening admissibility is complicated to prove. The system of [7] is for a a fragment that does not include full PLTL.

Here we aim at a more restricted result. We show that it is possible to limit the cut formulæ to a subset of the closure of the sequent whose validity is to be proved. To achieve that, we propose a set of rules with a cut rule that is restricted to something very similar to *Fischer-Ladner* cuts [3] and prove completeness with a slightly adapted version of the tableau method of [2].

The paper is organized as follows: the syntax and semantics of the language are presented in section 2. The tableau method is sketched in section 3 and the corresponding algorithm for proving satisfiability in section 4. In section 5 the sequent system is presented and its soundness proved. The completeness proof is in section 6. Section 8 contins the conclusions.

2 Syntax and Semantics

We will deal with the unary fragment of Propositional Linear-Time Temporal Logic (henceforth PLTL.) [1, **?**] Thus, the operators *release* and *until* will not be considered. We will use positive form, i.e., only atomic propositions will be negated. Negation of more complex formulæ will be obtained with their corresponding duals. We start from a countable set of atomic propositions Π whose elements will be denoted by p.
The language is given by the following grammar:

$$\varphi \ ::= \ p \mid \neg p \ \mid \varphi \vee \varphi \ \mid \varphi \wedge \varphi \ \mid \ \bigcirc \varphi \mid \Box\varphi \ \mid \Diamond\varphi$$

We will use $\top$ as an abbreviation of $p \vee \neg p$ and $\bot$ as an abbreviation of $p \wedge \neg p$. We will use also $\varphi \Rightarrow \psi$ as an abbreviation of $\neg\varphi \vee \psi$. The negation of a formula is defined as usual:

$$\neg\neg p \equiv p \qquad \neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi \qquad \neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$$
$$\neg \bigcirc \varphi \equiv \bigcirc\neg\varphi \qquad \neg\Box\varphi \equiv \Diamond\neg\varphi \qquad \neg\Diamond\varphi \equiv \Box\neg\varphi$$

Note that this implies that $\neg\top \equiv \bot$. The following definitions will be useful later.

Length and subformulæ of a formula The *length* $|\varphi|$ of a formula φ and the collection $\mathsf{Suf}(\varphi)$ of its *subformulæ* are inductively defined as follows:

- $|p| = 1$ and $\mathsf{Suf}(p) = \{p\}$
- if φ is $\bigcirc\alpha$, $\Diamond\alpha$, or $\Box\alpha$, then $|\varphi| = 1 + |\alpha|$ and $\mathsf{Suf}(\varphi) = \{\varphi\} \cup \mathsf{Suf}(\alpha)$
- if φ is $\alpha \vee \beta$ or $\alpha \wedge \beta$, then $|\varphi| = 1 + |\alpha| + |\beta|$ and $\mathsf{Suf}(\varphi) = \{\varphi\} \cup \mathsf{Suf}(\alpha) \cup \mathsf{Suf}(\beta)$

Lemma 2.1. *For any formula φ, $\mathsf{Suf}(\varphi) \leq |\varphi|$.*

Proof *Structural induction on φ.*
Base case: *if $\varphi = p$, then $|\mathsf{Suf}(p)| = 1 = |p|$*
Induction step: *if $\varphi = \neg\alpha$, $\varphi = \bigcirc\alpha$, $\varphi = \Box\alpha$, or $\varphi = \Diamond\alpha$, then*

$$|\mathsf{Suf}(\varphi)| = 1 + |\mathsf{Suf}(\alpha)| \leq_{IH} 1 + |\alpha| = |\varphi|$$

If $\varphi = \alpha \vee \beta$ or $\varphi = \alpha \wedge \beta$, then

$$|\mathsf{Suf}(\varphi)| = 1 + |\mathsf{Suf}(\alpha)| + |\mathsf{Suf}(\beta)| \leq_{IH} 1 + |\alpha| + |\beta| = |\varphi|$$

Closure of a formula The *closure* $\mathsf{Cl}(\varphi)$ of a formula φ is the union of all its subformulæ and their negations, i.e.

$$\mathsf{Cl}(\varphi) = \mathsf{Suf}(\varphi) \cup \{\neg\alpha \mid \alpha \in \mathsf{Suf}(\varphi)\}$$

As a corollary of the last definition and lemma 2.1, we have that $|\mathsf{Cl}(\varphi)| \leq 2 \cdot |\varphi|$.

Fischer-Ladner Closure The *Fischer-Ladner Closure* $\mathsf{FL}(\varphi)$ of a formula φ is defined as

$$\mathsf{FL}(\varphi) = \mathsf{Cl}(\varphi) \cup \{\bigcirc\alpha \mid \alpha \in \mathsf{Cl}(\varphi)\}$$

Note that this is not exactly the usual definition of the Fischer-Ladner closure, since we introduce the prefix $\bigcirc$. As a corollary of this definition and lemma 2.1, we have that $|\mathsf{FL}(\varphi)| \leq 2 \cdot |\mathsf{Cl}(\varphi)| \leq 4 \cdot |\varphi|$. The semantics of PLTL is given by infinite sequences of states. It is defined next.

Computation, Interpretation, Model Let S be a non-empty set. Then

- A *computation* over S is a function $\mathcal{C} : \mathbb{N} \mapsto S$.
- An *interpretation* over S is a function $\mathcal{I} : \Pi \mapsto 2^S$.
- A *model* for PLTL is a triple $\mathfrak{M} = (S, \mathcal{C}, \mathcal{I})$ where S is a non-empty set, $\mathcal{C}$ is a computation over S, and $\mathcal{I}$ is an interpretation over S.

The set S is called the set of *states*. A computation over S is therefore an infinite sequence of states.

Satisfiability Relation Let $\mathfrak{M} = (S, \mathcal{C}, \mathcal{I})$ be a model for PLTL. The *satisfiability relation* $(\mathfrak{M}, i) \models \varphi$ for any $i \in \mathbb{N}$ and any formula φ is inductively defined as follows:

- $(\mathfrak{M}, i) \models p$ if and only if $\mathcal{C}(i) \in \mathcal{I}(p)$.
- $(\mathfrak{M}, i) \models \neg\varphi$ if and only if $(\mathfrak{M}, i) \not\models \varphi$.
- $(\mathfrak{M}, i) \models \varphi \vee \psi$ if and only if $(\mathfrak{M}, i) \models \varphi$ or $(\mathfrak{M}, i) \models \psi$.
- $(\mathfrak{M}, i) \models \varphi \wedge \psi$ if and only if $(\mathfrak{M}, i) \models \varphi$ and $(\mathfrak{M}, i) \models \psi$.
- $(\mathfrak{M}, i) \models \bigcirc\varphi$ if and only if $(\mathfrak{M}, i+1) \models \varphi$.
- $(\mathfrak{M}, i) \models \Box\varphi$ if and only if for all $j \geq i$ it is the case that $(\mathfrak{M}, j) \models \varphi$.
- $(\mathfrak{M}, i) \models \Diamond\varphi$ if and only if for some $j \geq i$ it is the case that $(\mathfrak{M}, j) \models \varphi$.

Definition A model $\mathfrak{M}$ satisfies φ if $(\mathfrak{M}, 0) \models \varphi$. A formula φ is *satisfiable* if there exists some model $\mathfrak{M}$ which satisfies it. A formula φ is *valid* (denoted $\models \varphi$) if it is satisfied in all models.

3 A Tableau for PLTL

Throughout this section φ will be arbitrary but fixed. We follow essentially the exposition of [1, 2, **?**].

φ-atoms A *φ-atom* is a set $A \subset \mathrm{Cl}(\varphi)$ such that for all $\alpha, \beta_1 \vee \beta_2, \Diamond\gamma \in \mathrm{Cl}(\varphi)$, it satisfies the following properties:

1. $\alpha \in A$ i and only if $\neg\alpha \notin A$.
2. $\beta_1 \vee \beta_2 \in A$ if and only if $\beta_1 \in A$ or $\beta_2 \in A$.
3. if $\gamma \in A$ then $\Diamond\gamma \in A$.

Observe that for all $\Box\alpha, \beta_1 \wedge \beta_2 \in \mathrm{Cl}(\varphi)$, we have:

- If $\Box\alpha \in A$, then $\alpha \in A$:
  $\Box\alpha \in A \Leftrightarrow \neg\Box\alpha = \Diamond\neg\alpha \notin \alpha \Rightarrow \neg\alpha \notin A \Leftrightarrow \alpha \in A$.
- $\beta_1 \wedge \beta_2$ if and only if $\beta_1 \in A$ and $\beta_2 \in A$:
  $\beta_1 \wedge \beta_2 \in A \Leftrightarrow \neg(\beta_1 \wedge \beta_2) = \neg\alpha \vee \neg\beta \notin A \Leftrightarrow \neg\beta_1, \neg\beta_2 \notin A \Leftrightarrow \beta_1, \beta_2 \in A$.

We denote the set of all φ-atoms by $\mathcal{A}_\varphi$.

φ-successors Let $A, B \in \mathcal{A}_\varphi$. Then B is a *φ-successor* of A if for all $\bigcirc\alpha, \Diamond\beta \in \mathrm{Cl}(\varphi)$ we have:

1. $\bigcirc\alpha \in A$ if and only if $\alpha \in B$.
2. If $\Diamond\beta, \neg\beta \in A$, then $\Diamond\beta \in B$.
3. If $\Diamond\beta \in B$, then $\Diamond\beta \in A$.

Observe that if $\Box\gamma \in A$ and B is a φ-successor of A, then $\Box\gamma \in B$:
$\Box\gamma \in A \Leftrightarrow \neg\Box\gamma = \Diamond\neg\gamma \notin A \Rightarrow \Diamond\neg\gamma \notin B \Leftrightarrow \neg\Diamond\neg\gamma = \Box\neg\neg\gamma = \Box\gamma \in B$.
We will denote by $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$ the graph whose nodes are all the φ-atoms and whose edges are defined by the binary relation $\mathcal{R}_\varphi$ defined by $(A, B) \in \mathcal{R}_\varphi$ if and only if B is a φ-successor of A. If there is a finite path within $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$ from A to B (including a path of length 0), we say that B is a *descendant* of A.

Lemma 3.1. *Let $A_0, A_1, \ldots, A_n$ with $n \geq 0$ be a finite path in the graph $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$. Then*

1. *If $\Diamond\alpha \in A_0$ and $\neg\alpha \in A_i$ for all i, $0 \leq i \leq n-1$, then $\Diamond\alpha \in A_j$ for all j, $0 \leq j \leq n$.*

2. *If $\alpha \in A_n$, then $\Diamond\alpha \in A_0$.*

Proof *Part 1. Induction on the length of the path.*
Base case*: $n = 0$. Then part 1 holds by assumption.*
Induction step*: assume that for some $k > 0$ (1) holds. Then $\Diamond\alpha \in A_k$. If $\neg\alpha \in A_k$, then by definition of the successor relation, $\Diamond\alpha \in A_{k+1}$.*
Part 2. We prove that $\Diamond\alpha \in A_{n-j}$ by induction on j.
Base case*: $j = 0$. Then part 2 holds by definition of φ-atoms.*
Induction step*: if (2) holds for some $j > 0$, then $\Diamond\alpha \in A_{k-j}$. Then by definition of the successor relation, $\Diamond\alpha \in A_{k-j-1} = A_{k-(j+1)}$*

Now we establish a relation between paths in the graph and models.

Pre-models

1. A *pre-model with respect to* φ is an infinite path in $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$.
2. A *pre-model for* φ is a pre-model $A_0, A_1, \ldots$ with respect to φ such that $\varphi \in A_0$.

Lemma 3.2. *Let $\mathfrak{M}$ be a model. Then, if we set*

$$A_i := \{\alpha \in \mathsf{Cl}(\varphi) \mid (\mathfrak{M}, i) \models \alpha\}$$

for any $i \in \mathbb{N}$. Then

1. *$A_0, A_1, \ldots$ is a pre-model with respect to φ.*
2. *If $\mathfrak{M}$ satisfies φ, then $A_0, A_1, \ldots$ is a pre-model for φ.*

Proof *First let us verify that each A_i is an atom. The first condition, $S \subset$ $\mathsf{Cl}(\varphi)$ is clearly fulfilled. Besides:*

- $\alpha \in A_i \Leftrightarrow (\mathfrak{M}, i) \models \alpha \Leftrightarrow (\mathfrak{M}, i) \not\models \neg\alpha \Leftrightarrow \neg\alpha \notin A_i$.
- $\alpha \in A_i$ *or* $\beta \in A_i \Leftrightarrow (\mathfrak{M}, i) \models \alpha$ *or* $(\mathfrak{M}, i) \models \beta \Leftrightarrow (\mathfrak{M}, i) \models \alpha \vee \beta \Leftrightarrow$ $\alpha \vee \beta \in A_i$
- $\alpha \in A_i \Leftrightarrow (\mathfrak{M}, i) \models \alpha \Leftrightarrow$ *for some* $j \geq i(\mathfrak{M}, j) \models \alpha \Leftrightarrow (\mathfrak{M}, i) \models$ $\Diamond\alpha \Leftrightarrow$
$\Diamond\alpha \in A_i$

Now let us verify that for any pair A_j, A_{j+1}, $(A_j, A_{j+1}) \in \mathcal{R}_\varphi$:

- $\bigcirc\alpha \in A_i \Leftrightarrow (\mathfrak{M}, i) \models \bigcirc\alpha \Leftrightarrow (\mathfrak{M}, i+1) \models \alpha \Leftrightarrow \alpha \in A_{i+1}$.

- *Assume $\Diamond\alpha \in A_i$ (1) and $\neg\alpha \in A_i$ (2). By (1), $(\mathfrak{M}, i) \models \bigcirc\alpha$ and therefore there is some $j \geq i$ such that $(\mathfrak{M}, j) \models \alpha$. By (2) $(\mathfrak{M}, i) \not\models \alpha$ and thus $j > i$. Thus, $j \geq i+1$ and therefore $(\mathfrak{M}, i+1) \models \Diamond\alpha$. Hence, $\Diamond\alpha \in A_{i+1}$.*
- *Assume $\Diamond\alpha \in A_{i+1}$. Thus, $(\mathfrak{M}, i+1) \models \Diamond\alpha$ and therefore there is some $j \geq i+1$ such that $(\mathfrak{M}, j) \models \alpha$. Hence, $j \geq i$ and $(\mathfrak{M}, i) \models \Diamond\alpha$. Thus $\Diamond\alpha \in A_i$*

Thus, $A_0, A_1, \ldots$ is a pre-model with respect to φ. If $\mathfrak{M}$ satisfies φ, then $\varphi \in A_0$. Thus, $A_0, A_1, \ldots$ is a pre-model for φ.

We need yet another definition before we establish a correspondence between models and infinite paths in the graph.

Fulfilling Paths 1. A pre-model $A_0, A_1, \ldots$ with respect to φ is a *fulfilling path* if for every $i \in \mathbb{N}$ and every formula $\Diamond\alpha \in A_i$ there exists some $j > i$ such that $\alpha \in A_j$.

2. A *fulfilling path for φ* is a pre-model for φ which is also a fulfilling path.

The following theorem is crucial for the satisfiability procedure.

Theorem 3.3. *A formula φ is satisfiable if and only if there exists a fulfilling path for φ.*

Proof *Let us assume first that the formula is satisfiable. Then by lemma 3.2 there exists a fulfilling path for φ.*
Assume now that there exists a fulfilling path for φ. Consider the computation $\mathcal{C} : \mathbb{N} \mapsto \{A_i \mid i \in \mathbb{N}\}$ defined by

$$\mathcal{C}(i) := A_i$$

We will consider also the interpretation $\mathcal{I} : \Pi \mapsto 2^{\{A_i \mid i \in \mathbb{N}\}}$ defined by

$$\mathcal{I}(p) := \{A_i \mid i \in \mathbb{N} \text{ and } p \in A_i\}$$

Now set $\mathfrak{M} := (\{A_i \mid i \in \mathbb{N}\}, \mathcal{C}, \mathcal{I})$. We will prove that $\mathfrak{M} \models \alpha$ for any formula $\alpha \in CL(\varphi)$ if and only if $\alpha \in A_i$. We do this by induction on the structure of α.
Base case*: if $\alpha = p$, then $(\mathfrak{M}, i) \models p$ if and only if $\mathcal{C}(i) \in \mathcal{I}(p)$ if and only if $p \in A_i$.*
Induction step.

- *Let $\alpha = \neg\beta$. Then $(\mathfrak{M}, i) \models \neg\beta \Leftrightarrow (\mathfrak{M}, i) \not\models \beta \Leftrightarrow \beta \notin A_i \Leftrightarrow \neg\beta \in A_i$.*

- *Let* $\alpha = \beta_1 \vee \beta_2 \Leftrightarrow (\mathfrak{M}, i) \models \beta_1 \vee \beta_2 \Leftrightarrow \beta_1 \in A_i$ *or* $\beta_2 \in A_2 \Leftrightarrow \beta_1 \vee \beta_2 \in A_i$*.*

- *Let* $\alpha = \beta_1 \wedge \beta_2$*. Then* $(\mathfrak{M}, i) \models \beta_1 \wedge \beta_2 \Leftrightarrow \beta_1 \in A_i$ *and* $\beta_2 \in A_2 \Leftrightarrow \beta_1 \wedge \beta_2 \in A_i$*.*

- *Let* $\alpha = \bigcirc\beta$*. Then* $(\mathfrak{M}, i) \models \bigcirc\beta \Leftrightarrow (\mathfrak{M}, i+1) \models \beta \Leftrightarrow \beta \in A_{i+1} \Leftrightarrow \bigcirc\beta \in A_i$*.*

- *Let* $\alpha = \Diamond\beta$*. Then* $(\mathfrak{M}, i) \models \Box\beta \Leftrightarrow$ *for some* $j \geq i, (\mathfrak{M}, j) \models \beta \Leftrightarrow \beta \in A_j \Leftrightarrow \Diamond\beta \in A_i$*. The last step is by lemma 3.1.*

- *Let* $\alpha = \Box\beta$*. Then* $(\mathfrak{M}, i) \models \Box\beta \Leftrightarrow$ *for all* $j \geq i, (\mathfrak{M}, j) \models \beta \Leftrightarrow \beta \in A_j$*. Since the path is fulfilling,* $\Diamond\neg\beta \notin A_i \Leftrightarrow \neg\Diamond\neg\beta = \Box\neg\neg\beta = \Box\beta \in A_i$*.*

Besides, since $A_0, A_1, \ldots$ *is a fulfilling path for* φ*, then* $\varphi \in A_0$ *and therefore* $(\mathfrak{M}, 0) \models \varphi$*.*

The previous theorem states that it is enough to find a fulfilling path for φ to determine its satisfiability. Some definitions on graphs will be necessary before we go on.

Definition Let $\mathcal{G} \subseteq \mathcal{A}_\varphi$. Then

1. The *subgraph of* $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$ *generated by* $\mathcal{G}$ is the graph whose nodes are the elements of $\mathcal{G}$ and whose edges are the edges of $\mathcal{R}_\varphi$ which connect elements of $\mathcal{G}$.

2. The subgraph generated by $\mathcal{G}$ is *strongly connected* if for any two different nodes $A, B \in \mathcal{G}$, there exists a path in $\mathcal{G}$ leading from A to B.

3. A strongly connected subgraph is *maximally strongly connected* if it is not properly contained in another strongly connected subgraph.

We will use for simplicity the notation $\mathcal{G}$ to refer both to a subset of $\mathcal{A}_\varphi$ and to the subgraph generated by it.
Assume that $\mathcal{P}$ is a pre-model with respect to φ. We denote by $\mathsf{Inf}(\mathcal{P})$ the set of elements of $\mathcal{P}$ which appear infinitely often in $\mathcal{P}$. Observe that the set $\mathsf{Inf}(\varphi)$ is strongly connected.

Definition Let $\mathcal{G} \subseteq \mathcal{A}_\varphi$ be strongly connected. Then $\mathcal{G}$ is called *self-fulfilling* if $\mathcal{G} \neq \emptyset$ and for any $A \in \mathcal{G}$ and every formula $\Diamond\alpha \in A$, there exists a $B \in \mathcal{G}$ such that $\alpha \in B$.

Lemma 3.4. *Let $\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{A}_\varphi$ be two strongly connected subgraphs. Then if $\mathcal{G}$ is self-fulfilling, so is $\mathcal{G}'$.*

Proof *Let $A \in \mathcal{G}$ b such that $\Diamond\alpha \in A$. If $A \in \mathcal{G}$ we are done, since $\mathcal{G}'$ is self fulfilling and thus there is some $B \in \mathcal{G}$ such that $\alpha \in B$. Assume therefore that $A \in \mathcal{G}' \backslash \mathcal{G}$. Since $\mathcal{G}'$ is strongly connected, there is a path $A, C_1, \ldots, C_n, B$ to an atom $B \in \mathcal{G}$. Since $\diamond\alpha \in A$, by lemma 3.1 either $\alpha \in C_i$ for some i, or $\alpha \in B$, or $\Diamond\alpha \in B$. In the first two cases we are done. In the third case, since $\mathcal{G}$ is self-fulfilling, there is a $C \in \mathcal{G}$ such that $\alpha \in C$. Since $\mathcal{G}$ is strongly connected, there is a path from B to C and thus there is a path from A to C.*

Yet another definition:

Definition Let $\mathcal{G}$ be a maximally strongly connected subgraph of $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$. Then $\mathcal{G}$ is *useless* if one of the following conditions holds:

1. $\mathcal{G}$ is not reachable from any atom of $\mathcal{A}_\varphi$ containing φ, or
2. $\mathcal{G}$ contains no outgoing edges and is not self-fulfilling.

The following lemma states the basis of the decision procedure.

Lemma 3.5. *Let $\mathcal{G}$ be useless subgraph of $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$ and let $\mathcal{H}$ b an arbitrary subgraph of $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$. Then an infinite path in $\mathcal{H}$ is a fulfilling path for φ in $\mathcal{H}$ if and only if it is a fulfilling path for φ in $\mathcal{H} \backslash \mathcal{G}$.*

Proof *If $\mathcal{G}$ is useless because of the first condition, the path can never reach $\mathcal{G}$. Therefore, $\mathcal{P}$ is a fulfilling path for φ in $\mathcal{H}$ if and only if it is a fulfilling path for φ in $\mathcal{H} \backslash \mathcal{G}$.*
If $\mathcal{G}$ is useless because of the second condition, and $\mathcal{P}$ reaches $\mathcal{G}$, it has to stay there. Since the number of atoms is finite, any infinite path eventually remains in a strongly connected subgraph of $(\mathcal{A}_\varphi, \mathcal{R}_\varphi)$. According to lemma 3.4 it has to be self-fulfilling, thus contradicting the assumption. Hence, $\mathcal{P}$ is in $\mathcal{H}$ if and only if it is in $\mathcal{H} \backslash \mathcal{G}$.

4 Satisfiability-Checking

Let us consider the following algorithm.

$(\mathcal{A}_0, \mathcal{R}_0) := (\mathcal{A}_\varphi, \mathcal{R}_\varphi)$;
$i := 0$;
while $(\mathcal{A}_i, \mathcal{R}_i)$ contains some useless subgraph **do**
begin
    choose a useless graph $\mathcal{G}$ of $(\mathcal{A}_i, \mathcal{R}_i)$;
    $\mathcal{A}_{i+1} := \mathcal{A}_i \backslash \mathcal{G}$;
    $\mathcal{R}_{i+1} := \mathcal{R}_i \cap (\mathcal{A}_{i+1} \times \mathcal{A}_{i+1})$;
    $i := i + 1$;
end;
$\mathcal{A}_F := \mathcal{A}_i$;
if $\mathcal{A}_F \neq \emptyset$ **then**
    report `success`
else
    report `failure`;

Theorem 4.1. *A formula φ is satisfiable if and only if the algorithm reports* `success`*.*

Proof *Assume φ is satisfiable. By theorem 3.3, there exists a fulfilling path for φ. By the construction of the algorithm and lemma 3.5, this path must be in $\mathcal{A}_F$ and thus it cannot be empty. Hence,* `success` *is reported.*
Assume now that $\mathcal{A}_F \neq \emptyset$. Thus, there must be a finite path $A_0, \ldots, A_k$ such that $\varphi \in A_0$ and A_k belongs to a self-fulfilling maximal strongly connected subgraph. Thus there exists a fulfilling path for φ and therefore φ is satisfiable.

5 A Sequent System for PLTL

The sequent system $S_0(\varphi)$ is given by the following rules.

$$\text{id}\ \frac{}{\Gamma, p, \overline{p}} \qquad \vee\ \frac{\Gamma, \alpha, \beta}{\Gamma, \alpha \vee \beta} \qquad \wedge\ \frac{\Gamma, \alpha \quad \Gamma, \beta}{\Gamma, \alpha \wedge \beta} \qquad \bigcirc\ \frac{\Gamma}{\bigcirc\Gamma, \Sigma}$$

$$\Box\ \frac{\Gamma, \alpha \quad \Gamma, \bigcirc\Box\alpha}{\Gamma, \Box\alpha} \qquad \Diamond\ \frac{\Gamma, \alpha, \bigcirc\Diamond\alpha}{\Gamma, \Diamond\alpha} \qquad \text{ind}\ \frac{\neg\alpha, \bigcirc\alpha \quad \neg\alpha, \beta}{\neg\alpha, \Box\beta, \Sigma}$$

$$\text{a-cut}\ \frac{\Gamma, \alpha \quad \Gamma, \neg\alpha}{\Gamma}\ \alpha \in \mathsf{FL}(\varphi) \qquad \Box\Diamond\ \frac{\Gamma, \alpha}{\Diamond\Gamma, \Box\alpha, \Sigma}$$

Figure 1: The system S_0.

A sequent Γ is a collection of formulæ. It is interpreted as the disjunction of all the formulæ it consists of.
Note that this system has an *analytical cut*. This means that cuts are only allowed with formulæ and that these formulæ are parameterised by φ. If we lift this restriction we get a normal system with arbitrary cuts, which is known to be sound and complete. In our case, the only cut-formulæ that will be used are those in the Fischer-Ladner closure of φ.

Theorem 5.1. *The system $S_0(\varphi)$ is sound.*

Proof *We consider the different rules separately.*
(id) *Assume $\not\models \Gamma, p, \neg p$. Thus there is some model $\mathfrak{M} = (\mathcal{C}, \mathcal{I}, S)$ such that for $i \in \mathbb{N}$, $(\mathfrak{M}, i) \models \neg\Gamma \wedge p \wedge \neg p$. Thus, $(\mathfrak{M}, i) \models p$ and $(\mathfrak{M}, i) \models \neg p$. Hence, $\mathcal{C}(i) \in \mathcal{I}(p)$ and $\mathcal{C}(i) \notin \mathcal{I}(p)$, which is a contradiction.*
$(\vee)$ *Assume $\models \Gamma, \alpha, \beta$ (1) but $\not\models \Gamma\alpha \vee \beta$. Then there is some model $\mathfrak{M}$ such that, for some $i \in \mathbb{N}$, it holds that $(\mathfrak{M}, i) \models \neg\Gamma \wedge \neg\alpha \wedge \neg\beta$, which contradicts (1).*
$(\vee)$ *Assume $\models \Gamma, \alpha$ (1) and $\models \Gamma, \beta$ (2) but $\not\models \Gamma, \alpha \wedge \beta$. Then there is some model $\mathfrak{M}$ such that, for some $i \in \mathbb{N}$, it holds that $(\mathfrak{M}, i) \models \neg\Gamma \wedge \neg(\alpha \wedge \beta)$ (3). From (1) and (3) we get $(\mathfrak{M}, i) \models \alpha$ (4) and from (2) and (3) we get $(\mathfrak{M}, i) \models \beta)$ (5). Hence, from (4) and (5) we get $(\mathfrak{M}, i) \models \alpha \wedge \beta$, which contradicts (3).*
$(\bigcirc)$ *Assume $\models \Gamma$ (1) but $\not\models \bigcirc\Gamma, \Sigma$. Then there is some model $\mathfrak{M}$ such that, for some $i \in \mathbb{N}$, it holds that $(\mathfrak{M}, i) \models \neg \bigcirc \Gamma \wedge \neg\Sigma$. Hence, $(\mathfrak{M}, i+1) \models \neg\Gamma$, which contradicts (1).*
$(\Box)$ *Assume $\models \Gamma, \alpha$ (1) and $\models \Gamma, \bigcirc\Box\alpha$ (2) but $\not\models \Gamma, \Box\alpha$. Thus there is some model $\mathfrak{M}$ such that, for some $i \in \mathbb{N}$, it holds that $(\mathfrak{M}, i) \models \neg\Gamma \wedge \neg\Box\alpha$*

(3). From (1) and (3) we get $(\mathfrak{M}, i) \models \alpha$ *(4) and from (3) and (4) we get* $(\mathfrak{M}, i) \models \neg\Gamma \wedge \bigcirc\neg\Box\alpha$*, which contradicts (2).*
$(\Diamond)$ *Assume* $\models \Gamma, \alpha, \bigcirc\Diamond\alpha$ *(1) but* $\not\models \Gamma, \Diamond\alpha$*. Thus, there is some model* $\mathfrak{M}$ *such that, for some* $i \in \mathbb{N}$*, it holds that* $(\mathfrak{M}, i) \models \neg\Gamma \wedge \neg\Diamond\alpha$*. Hence, for all* $j \geq i$*,* $(\mathfrak{M}, j) \models \neg\alpha$ *(2). Besides, from (1) follows* $(\mathfrak{M}, i+1) \models \Diamond\alpha$*. Hence, there is some* $j > i$ *such that* $(\mathfrak{M}, j) \models \alpha$*, which contradicts (2).*
(ind) *Assume* $\models \neg\alpha, \bigcirc\alpha$ *(1) and* $\models \neg\alpha, \beta$ *(2) but* $\not\models \neg\alpha, \Box\beta, \Sigma$*. Thus, there is a model* $\mathfrak{M}$ *such that, for some* $i \in \mathbb{N}$*, it holds that* $(\mathfrak{M}, i) \models \alpha \wedge \neg\Box\beta \wedge \neg\Sigma$ *(3). Hence, there is some* $j \geq i$ *such that* $(\mathfrak{M}, j) \models \neg\beta$ *(4). We prove that for all* $k \geq i$*,* $(\mathfrak{M}, k) \models \alpha$ *and* $(\mathfrak{M}, k) \models \beta$*, which contradicts (4), since these imply* $(\mathfrak{M}, i) \models \Box\beta$*.*
The base case is $k = 0$*. Then (2) and (3) yield* $(\mathfrak{M}, i) \models \alpha$ *and* $(\mathfrak{M}, i) \models \beta$*. For the induction step, assume that for some* $(\mathfrak{M}, k) \models \alpha$ *(5) and* $(\mathfrak{M}, k) \models \beta$*. From (5) and (1) we get* $(\mathfrak{M}, k) \models \bigcirc\alpha$ *and thus* $(\mathfrak{M}, k+1) \models \alpha$ *(6). From (6) and (2) we get* $(\mathfrak{M}, k+1) \models \beta$*.*
(a-cut) *Assume* $\models \Gamma, \varphi$ *and* $\models \Gamma, \neg\varphi$ *but* $\not\models \Gamma$*. Thus, there is a model* $\mathfrak{M}$ *such that for some* $i \in \mathbb{N}$*,* $(\mathfrak{M}, i) \models \neg\Gamma$*. Thus,* $(\mathfrak{M}, i) \models \varphi$ *and* $(\mathfrak{M}, i) \models \neg\varphi$*, which is a contradiction.* $(\Box\Diamond)$ *Assume* $\models \Gamma, \varphi$ *(1) but* $\not\models \Diamond\Gamma, \Box\varphi, \Sigma$*. Thus, there is a model* $\mathfrak{M}$ *such that for some* $i \in \mathbb{N}$*,* $(\mathfrak{M}, i) \models \neg\Diamond\Gamma \wedge \neg\Box\varphi \wedge \neg\Sigma$ *(2). Hence, for all* $j \geq i$*,* $(\mathfrak{M}, j) \models \neg\Gamma$ *(3) and thus from (1) and (3) for all* $j \geq i$*,* $(\mathfrak{M}, j) \models \varphi$*. Hence,* $(\mathfrak{M}, i) \models \Box\varphi$*, which contradicts (2).*

6 Completeness of $S_0(\varphi)$

The aim of this section is to prove that if the algorithm yields an empty graph for a formula $\neg\varphi$, then this formula may be proven false in $S_0(\neg\varphi)$ or, equivalently, in $S_0(\varphi)$. Given an atom A, we will use the notation $\hat{A}$ for the conjunction of all elements of A, i.e. $\hat{A} := \bigwedge_{\alpha \in A}\{\alpha\}$. We will use some simplifications: we will write $\vdash$ instead of $S_0(\varphi) \vdash$ and we will use the rule $\vee$ in both directions. This is justified by lemma 7.1.

Lemma 6.1. *$\vdash \bigvee_{A \in \mathcal{A}_\varphi} \hat{A}$.*

Proof *Let $\mathsf{Cl}(\varphi) = \{\alpha_1, \ldots, \alpha_m, \neg\alpha_1, \ldots, \neg\alpha_m\}$. Then we get:*

(1.1)	$\vdash \alpha_1, \neg\alpha_1$	*lemma 7.2*
	$\ldots\ldots\ldots$	
$(1.\mathrm{m})$	$\vdash \alpha_m, \neg\alpha_m$	*lemma 7.2*
(2.1)	$\vdash \alpha_1 \vee \neg\alpha_1$	$\vee$ *on (1.1)*
	$\ldots\ldots\ldots$	
$(2.\mathrm{m})$	$\vdash \alpha_m \vee \neg\alpha_m$	$\vee$ *on (1.m)*
(3)	$\vdash (\alpha_1 \vee \neg\alpha_1) \wedge \ldots \wedge (\alpha_m \vee \alpha_m)$	$\wedge$ *on (2.1),..., (2.m)*

Applying m times lemma 7.5 on (3) we get something of the form

$$\vdash \hat{\Psi}_1 \vee \hat{\Psi}_2 \vee \ldots \vee \hat{\Psi}_{2^m} \qquad \text{(i)}$$

where for each Ψ_i and for each α_j, $1 \leq i \leq 2^m$, $1 \leq j \leq m$, $\alpha_j \in \Psi_i$ if and only if $\neg\alpha_j \notin \Psi_i$. In other words, each Ψ_i contains either α_j or $\neg\alpha_j$. It is easy to see that all atoms are contained in the set of the Ψ_i's, but there are also Ψ_i's which are not atoms. Assume that the first p elements are the atoms. The idea of the proof is to show that $S_0 \vdash \neg\hat{\Psi}_k$ whenever Ψ_k is not an atom (i.e., $k > p$) and then to cut away the elements that are not atoms. Assume Ψ_k is not an atom. Then, since $\alpha_j \in \Psi_k$ if and only if $\neg\alpha_j \notin \Psi_k$, we have the following possibilities:

- *$\beta_1 \vee \beta_2 \in \Psi_k$ but $\beta_1, \beta_2 \notin \Psi_k$ for some $\beta_1 \vee \beta_2 \in \mathsf{Cl}(\varphi)$. Then $\neg\hat{\Psi}_k = \neg\psi_1 \vee \ldots \vee \neg \vee \psi_q \vee \beta_1, \beta_2, \neg(\beta_1 \vee \beta_2)$. Thus*

(1)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \beta_1, \beta_2, \neg\beta_1$	*lemma 7.2*
(2)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \beta_1, \beta_2, \neg\beta_2$	*lemma 7.2*
(3)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \beta_1, \beta_2, \neg\beta_1 \wedge \neg\beta_2$	*$\wedge$ on (1), (2)*
(4)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \beta_1, \beta_2, \neg(\beta_1 \vee \beta_2)$	*def. neg. on (3)*
(5)	$\vdash \neg\psi_1 \vee \ldots \vee \neg\psi_q \vee \beta_1 \vee \beta_2 \vee \neg(\beta_1 \vee \beta_2)$	*$\vee$ on (4)*
(6)	$\vdash \neg\hat{\Psi}_k$	*def. $\hat{\Psi}_k$ on (6)*

- *$\beta_1 \in \Psi_k$ but $\beta_1 \vee \beta_2 \notin \Psi_k$ for some $\beta_1 \vee \beta_2 \in$ Cl(φ). Then $\neg\hat{\Psi}_k = \neg\psi_1 \vee \ldots \vee \neg \vee \psi_q \vee \neg\beta_1 \vee \beta_1 \vee \beta_2$. Thus*

(1)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \neg\beta_1, \beta_1, \beta_2$	*lemma 7.2*
(2)	$\vdash \neg\psi_1 \vee \ldots \vee \neg\psi_q \vee \beta_1 \vee \beta_2$	*$\vee$ on (1)*
(3)	$\vdash \neg\hat{\Psi}_k$	*def. $\hat{\Psi}_k$ on (2)*

- *$\beta \in \Psi_k$ but $\Diamond\beta \notin \Psi_k$ for some $\Diamond\beta \in$ Cl(φ). Then $\neg\hat{\Psi}_k = \neg\psi_1 \vee \ldots \vee \neg \vee \psi_q \vee \neg\beta \vee \Diamond\beta$. Thus*

(1)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \neg\beta, \beta, \bigcirc\Diamond\beta$	*lemma 7.2*
(2)	$\vdash \neg\psi_1, \ldots, \neg\psi_q, \neg\beta, \Diamond\beta$	*$\Diamond$ on (1)*
(3)	$\vdash \neg\psi_1 \vee \ldots \vee \neg\psi_q \vee \neg\beta \vee \Diamond\beta$	*$\vee$ on (2)*
(4)	$\vdash \neg\hat{\Psi}_k$	*def. $\hat{\Psi}_k$ on (3)*

Thus, we have

$$\vdash \neg\hat{\Psi}_{p+1} \wedge \ldots \wedge \neg\hat{\Psi}_{2^m} \qquad \textit{(ii)}$$

Since all elements of the conjunctions belong to Cl$(\varphi) \subset$ FL(φ)*, we may apply lemma 7.6 on (i) and (ii) and we get*

$$\vdash \hat{\Psi}_1 \vee \hat{\Psi}_2 \vee \ldots \vee \hat{\Psi}_p$$

Lemma 6.2. *For any formula $\alpha \in$ Cl(φ) and any atom $A \in \mathcal{A}_\varphi$, if $\alpha \notin A$ then $\vdash \alpha \Rightarrow \neg\hat{A}$.*

Proof *We have that $\neg\hat{A} = \neg\beta_1 \vee \ldots \vee \neg\beta_m \vee \alpha$. Thus*

(1)	$\vdash \neg\alpha, \neg\beta_1, \ldots, \neg\beta_q, \alpha$	*lemma 7.2*
(2)	$\vdash \neg\alpha \vee (\neg\beta_1 \vee \ldots \vee \neg\beta_m \vee \alpha$	*$\vee$ on (1)*
(3)	$\vdash \neg\alpha \vee \neg\hat{A}$	*def. $\hat{A}$ on (2)*
(4)	$\vdash \alpha \Rightarrow \neg\hat{A}$	*def. $\Rightarrow$ on (3)*

Lemma 6.3. *For all formulæ $\alpha \in \mathsf{Cl}(\varphi)$, and for all atoms $A \in \mathcal{A}_\varphi$, $\vdash \alpha \Rightarrow \bigvee_{\alpha\in A\in\mathcal{A}_0} \hat{A}$.*

Proof *Assume $B \in \mathcal{A}_0$ and $\alpha \notin B$. Thus, $\neg\hat{B} = \beta_1 \vee \ldots \vee \beta_n \vee \alpha$. Thus*

(1)	$\vdash \neg\alpha, \neg\beta_1, \ldots, \neg\beta_n, \alpha$	*lemma 7.2*
(2)	$\vdash \neg\alpha, (\neg\beta_1 \vee \ldots \vee \neg\beta_n \vee \alpha)$	*$\vee$ on (1)*
(3)	$\vdash \neg\alpha, \neg\hat{B}$	*def. $\hat{B}$ on (2)*
(4)	$\vdash \neg\alpha, \bigwedge_{\alpha\notin B\in\mathcal{A}_0} \neg\hat{B}$	*$\wedge$ on (4)*
(5)	$\vdash \bigvee_{A\in\mathcal{A}_0} \hat{A}$	*lemma 6.2*
(7)	$\vdash \neg\alpha, \bigvee_{A\in\mathcal{A}_0} \hat{A}$	*lemma 7.3 on (6)*
(8)	$\vdash \neg\alpha, \bigvee_{\alpha\in A\in\mathcal{A}_0\hat{A}}$	*lemma 7.6 on (4), (7)*
(9)	$\vdash \neg\alpha \vee \bigvee_{\alpha\in A\in\mathcal{A}_0\hat{A}}$	*$\vee$ on (8)*
(10)	$\vdash \alpha \Rightarrow \bigvee_{\alpha\in A\in\mathcal{A}_0\hat{A}}$	*def. of $\Rightarrow$ (9)*

The application of lemma 7.6 is possible in step (8) because we are working with φ-atoms, all whose elements belong to $\mathsf{Cl}(\varphi) \subset \mathsf{FL}(\varphi)$.

Lemma 6.4. *For every atom $A \in \mathcal{A}_0$, $\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{B\in\mathcal{A}_0} \hat{B}$.*

Proof

(1)	$\vdash \bigvee_{B\in\mathcal{A}_0} \hat{B}$	*lemma 6.1*
(2)	$\vdash \neg\hat{A}, \bigcirc \bigvee_{B\in\mathcal{A}_0} \hat{B}$	*$\bigcirc$ on (1)*
(3)	$\vdash \neg\hat{A} \vee \bigcirc \bigvee_{B\in\mathcal{A}_0} \hat{B}$	*$\vee$ on (2)*
(4)	$\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{B\in\mathcal{A}_0} \hat{B}$	*def. of $\Rightarrow$ on (3)*

Lemma 6.5. *For any two atoms $A, B \in \mathcal{A}_0$ such that $(A, B) \notin \mathcal{R}_0$, we have that $\vdash \hat{A} \Rightarrow \bigcirc\neg\hat{B}$.*

Proof *We consider the different reasons why $(A, B) \notin \mathcal{R}_0$.*

- *$\bigcirc\psi \in A$ and $\psi \notin B$. Thus, we have*

$$\begin{array}{rcll} \neg\hat{A} & = & \neg\alpha_1 \vee \ldots \vee \alpha_m \vee \bigcirc\neg\psi & \textit{(i)} \\ \neg\hat{B} & = & \neg\beta_1 \vee \ldots \vee \neg\beta_n \vee \psi & \textit{(ii)} \end{array}$$

Therefore

(1)	$\vdash \neg\psi, \neg\beta_1, \ldots, \neg\beta_n, \psi$	*lemma 7.2*
(2)	$\vdash \neg\psi, \neg\beta_1 \vee \ldots \vee \neg\beta_n \vee \psi$	$\vee$ *on (1)*
(3)	$\vdash \neg\psi, \neg\hat{B}$	*(ii) on (2)*
(4)	$\vdash \alpha_1, \ldots, \alpha_m \bigcirc \neg\psi, \bigcirc\neg\hat{B}$	$\bigcirc$ *on (3)*
(5)	$\vdash \alpha_1 \vee \ldots \vee \alpha_m \bigcirc \neg\psi \vee \bigcirc\neg\hat{B}$	$\vee$ *on (4)*
(6)	$\vdash \neg\hat{A} \vee \bigcirc\neg\hat{B}$	*(i) on (5)*
(7)	$\vdash \hat{A} \Rightarrow \bigcirc\neg\hat{B}$	*def. of* $\Rightarrow$ *on (6)*

- $\psi \in B$ *and* $\bigcirc\psi \notin A$. *This case is analogous to the previous one. Just take* $\xi = \neg\psi$.

- $\Diamond\psi, \neg\psi \in A$ *and* $\Diamond\psi \notin B$. *Thus,*

$$\begin{array}{rcll} \neg\hat{A} & = & \neg\alpha_1 \vee \ldots \vee \alpha_m \vee \psi \vee \Box\neg\psi & \textit{(iii)} \\ \neg\hat{B} & = & \neg\beta_1 \vee \ldots \vee \neg\beta_n \vee \Diamond\psi & \textit{(iv)} \end{array}$$

Therefore

(1)	$\vdash \psi, \neg\psi$	*lemma 7.2*
(2)	$\vdash \Box\neg\psi, \Diamond\psi, \neg\beta_1, \ldots, \neg\beta_n$	$\Box\Diamond$ *on (1)*
(3)	$\vdash \Box\neg\psi, \Diamond\psi \vee \neg\beta_1 \vee \ldots \vee \neg\beta_n$	$\vee$ *on (2)*
(4)	$\vdash \Box\neg\psi, \neg\hat{B}$	*(iv) on (3)*
(5)	$\vdash \neg\alpha_1, \ldots, \alpha_m, \bigcirc\Box\neg\psi, \bigcirc\neg\hat{B}$	$\bigcirc$ *on (4)*
(6)	$\vdash \alpha_1, \ldots, \alpha_m, \psi, \neg\psi, \bigcirc\neg\hat{B}$	*lemma 7.2*
(7)	$\vdash \neg\alpha_1, \ldots, \alpha_m, \Box\neg\psi, \psi, \bigcirc\neg\hat{B}$	$\Box$ *on (5), (6)*
(8)	$\vdash \neg\alpha_1 \vee \ldots \vee \alpha_m \vee \Box\neg\psi \vee \psi \vee \bigcirc\neg\hat{B}$	$\vee$ *on (7)*
(9)	$\vdash \neg\hat{A} \vee \bigcirc\neg\hat{B}$	*(iii) on (8)*
(10)	$\vdash \hat{A} \Rightarrow \bigcirc\neg\hat{B}$	*def. of* $\Rightarrow$ *on (9)*

- $\Diamond\psi \in B$, $\Diamond\psi \notin A$. *Thus*

$$\neg\hat{A} = \neg\alpha_1 \vee \ldots \vee \alpha_m \vee \Diamond\psi \quad \textit{(v)}$$
$$\neg\hat{B} = \neg\beta_1 \vee \ldots \vee \neg\beta_n \vee \Box\neg\psi \quad \textit{(vi)}$$

Therefore

(1)	$\vdash \psi, \neg\psi$	*lemma 7.2*
(2)	$\vdash \Box\neg\psi, \Diamond\psi, \neg\beta_1, \ldots, \neg\beta_n$	$\Box\Diamond$ *on (1)*
(3)	$\vdash \Diamond\psi, \Box\neg\psi \vee \neg\beta_1 \vee \ldots \vee \neg\beta_n$	$\vee$ *on (2)*
(4)	$\vdash \Diamond\psi, \neg\hat{B}$	*(vi) on (3)*
(5)	$\vdash \neg\alpha_1, \ldots, \alpha_m, \psi, \bigcirc\Diamond\psi, \bigcirc\neg\hat{B}$	$\bigcirc$ *on (4)*
(6)	$\vdash \alpha_1, \ldots, \alpha_m, \Diamond\psi, \bigcirc\neg\hat{B}$	$\Diamond$ *on (5)*
(7)	$\vdash \neg\alpha_1 \vee \ldots \vee \alpha_m \vee \Diamond\psi \vee \bigcirc\neg\hat{B}$	$\vee$ *on (6)*
(8)	$\vdash \neg\hat{A} \vee \bigcirc\neg\hat{B}$	*(v) on (7)*
(9)	$\vdash \hat{A} \Rightarrow \bigcirc\neg\hat{B}$	*def. of* $\Rightarrow$ *on (8)*

Lemma 6.6. *For all atoms* $A, B \in \mathcal{A}_0$*,* $\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{(A,B)\in\mathcal{R}_0} \hat{B}$*.*

Proof *We denote by* C *the atoms such that* $(A, C) \notin \mathcal{R}_0$*.*

(1)	$\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{B\in\mathcal{A}_0} \hat{B}$	*lemma 6.4*
(2)	$\vdash \hat{A} \Rightarrow \bigcirc\neg\hat{C}$	*lemma 6.5*
(3)	$\vdash \hat{A} \Rightarrow \bigvee_{B\in\mathcal{A}_0} \bigcirc\hat{B}$	*lemma 7.7 on (1)*
(4)	$\vdash \hat{A} \Rightarrow \bigwedge_{(A,B)\notin\mathcal{R}_0} \bigcirc\neg\hat{C}$	$\wedge$ *on (2)*
(5)	$\vdash \hat{A} \Rightarrow \bigvee_{(A,B)\in\mathcal{R}_0} \bigcirc\hat{B}$	*lemma 7.6 on (3), (4)*

The use of lemma 7.6 in step (5) is justified because we want to cut formulæ which are of the form $\bigcirc\hat{B}$ *and* $\bigcirc\neg\hat{B}$*, where* B *are* φ*-atoms. Since the next-operator* $\bigcirc$ *can be pushed inside conjunction and disjunctions (lemma 7.7), we have that the cut-formulæ are of the form* $\bigcirc\neg\psi$ *and* $\bigcirc\psi$ *with* $\psi, \neg\psi \in \mathsf{Cl}(\varphi)$*. Therefore all cut- formulæ belong to* $\mathsf{FL}(\varphi)$*.*

Now we "lift" the preceding results from $(\mathcal{A}_0, \mathcal{R}_0)$ to $(\mathcal{A}_i, \mathcal{R}_i)$ for any $i \geq 0$. We will use the notation $\mathcal{R}^*$ for the reflexive and transitive closure of the relation $\mathcal{R}$.

Theorem 6.7. *For all* $i \geq 0$*, we have:*

(I) $\vdash \alpha \Rightarrow \bigvee_{\alpha\in A\in\mathcal{A}_i} \hat{A}$

(II) $\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{(A,B)\in\mathcal{R}_i} \hat{B}$

(III) $\vdash \hat{A} \Rightarrow \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_i} \neg\hat{C}$

Proof *Induction on* i.
Base case*:* $i = 0$*.*
Then **(I)** *is lemma 6.2,* **(II)** *is lemma 6.4 and for* **(III)** *we have, given two atoms* A *and* C *such that* $(A, C) \notin \mathcal{R}_0$*:*

(1)	$\vdash \hat{A} \Rightarrow \bigcirc\neg\hat{C}$	*lemma 6.5*
(2)	$\vdash \neg\hat{A} \vee \bigcirc\neg\hat{C}$	*def. of* $\Rightarrow$ *on (1)*
(3)	$\vdash \neg\hat{A}, \bigcirc\neg\hat{C}$	$\vee$ *on (2)*
(4)	$\vdash \neg\hat{A}, \bigwedge_{(A,C)\notin\mathcal{R}_0} \bigcirc\neg\hat{C}$	$\wedge$ *on (3)*
(5)	$\vdash \neg\hat{A}, \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_0} \neg\hat{C}$	*lemma 7.7 on (4)*
(6)	$\vdash \neg\hat{A} \vee \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_0} \neg\hat{C}$	$\vee$ *on (5)*
(7)	$\vdash \hat{A} \Rightarrow \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_0} \neg\hat{C}$	*def. of* $\Rightarrow$ *on (6)*

Induction step*:* $i \mapsto i + 1$
Let $\mathcal{C} = \mathcal{A}_i \backslash \mathcal{A}_{i+1}$*. Thus, either* $\mathcal{C}$ *is not reachable from any atom containing* φ *(***case 1***) or it is a strongly connected subgraph that is not self-fulfilling and has no outgoing edges (***case 2***.) We will use* C *to denote atoms belonging to* $\mathcal{C}$*.*
We consider first **case 1***. Note that for all atoms* $C \in \mathcal{C}$*,* $\varphi \notin C$*. For* **(I)** *we have:*

(1)	$\vdash \varphi \Rightarrow \bigvee_{\varphi\in A\in\mathcal{A}_i} \hat{A}$	*IH*
(2)	$\vdash \varphi \Rightarrow (\bigvee_{\varphi\in A\in\mathcal{A}_{i+1}} \hat{A} \vee \bigvee_{\varphi\in A\in\mathcal{A}_i\backslash\mathcal{A}_{i+1}} \hat{A})$	*disj. of (1)*
(3)	$\vdash \varphi \rightarrow \neg\hat{C}$	*lemma 6.2*
(4)	$\vdash \varphi \Rightarrow \neg\hat{A}$ *if* $A \in \mathcal{A}_i\backslash\mathcal{A}_{i+1}$	*because of (3)*
(5)	$\vdash \varphi \Rightarrow \bigwedge_{A\in\mathcal{A}_i\backslash\mathcal{A}_{i+1}} \neg\hat{A}$	$\wedge$ *on (4)*
(6)	$\vdash \varphi \Rightarrow \bigvee_{\varphi\in A\in\mathcal{A}_{i+1}} \hat{A}$	*lemma 7.6 on (2), (5)*

Use of lemma 7.6 in step (6) is allowed because we are cutting φ*-atoms, whose elements are in* $\mathsf{Cl}(\varphi) \subset \mathsf{FL}(\varphi)$*.*
For **(II)***, observe that for any atoms* $A \in \mathcal{A}_{i+1}$ *and* $C \in \mathcal{C}$*,* $(A, C) \notin \mathcal{R}_i$*. By induction hypothesis we have that* $\vdash \hat{A} \Rightarrow \bigvee_{(A,B)\in\mathcal{R}_i} \bigcirc\hat{B}$*, which may be written as* $\vdash \hat{A} \Rightarrow (\bigvee_{(A,B)\in\mathcal{R}_{i+1}} \bigcirc\hat{B} \vee \bigvee_{(A,B)\in\mathcal{R}_i\backslash\mathcal{R}_{i+1}} \bigcirc\hat{B})$*. This expression boils down to* $\vdash \hat{A} \Rightarrow (\bigvee_{(A,B)\in\mathcal{R}_{i+1}} \bigcirc\hat{B} \vee \bot)$*, since the second disjunction is empty. The result follows by lemma 7.8.*

For **(III)***, we have by induction hypothesis that* $\vdash \hat{A} \Rightarrow \bigwedge_{(A,B)\notin\mathcal{R}_i} \bigcirc\neg\hat{B}$*. Besides, for all* $C \in \mathcal{C}$*, we have that for all* $A \in \mathcal{A}_{i+1}$*,* $(A, C) \notin \mathcal{R}_i$*. Hence, in the passage from* i *to* $i+1$ *the conjunction* $\bigwedge_{(A,B)\notin\mathcal{R}_{i+1}} \hat{C}$ *has no new element with respect to* $\bigwedge_{(A,B)\notin\mathcal{R}_i} \hat{C}$*, since for all elements* C *that we eliminate it holds that* $(A, C) \notin \mathcal{R}_i$*. Therefore,* $\vdash \hat{A} \Rightarrow \bigwedge_{(A,B)\notin\mathcal{R}_{i+1}} \bigcirc\neg\hat{B}$*.*

Now we consider the **case 2***, when* $\mathcal{C}$ *is a strongly connected subgraph that is not self-fulfilling and has not outgoing edges. As before, we will use* C *to denote atoms belonging to* $\mathcal{C}$*.*

Let D *be an arbitrary atom such that* $(C, D) \in \mathcal{R}_i^*$*. Then* $D \in \mathcal{C}$*, since* $\mathcal{C}$ *is strongly connected and has no outgoing edges. Thus, if* $(D, E) \in \mathcal{R}_i$*, it is clear that* $(C, E) \in \mathcal{R}_i^*$*. Besides, since* $\mathcal{C}$ *is not self-fulfilling, there must be some* $\alpha \in \mathrm{Cl}(\varphi)$ *such that, for any atom,* $C \in \mathcal{C}$*, we have*

$$\neg\alpha \in C \qquad \textit{(i)}$$
$$\Diamond\alpha \in C \qquad \textit{(ii)}$$

Hence

(1)	$\vdash \bigvee_{(D,E)\in\mathcal{R}_i} \hat{E} \Rightarrow \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}$	*lemma 7.9*
(2)	$\vdash \bigcirc \bigvee_{(D,E)\in\mathcal{R}_i} \hat{E} \Rightarrow \bigcirc \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}$	*lemma 7.10 on (1)*
(3)	$\vdash \bigvee_{(D,E)\in\mathcal{R}_i} \bigcirc\hat{E} \Rightarrow \bigcirc \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}$	*lemma 7.7 on (1)*
(4)	$\vdash \hat{D} \Rightarrow \bigcirc \bigvee_{(D,E)\in\mathcal{R}_i} \hat{E}$	**(II)** *IH*
(5)	$\vdash \hat{D} \Rightarrow \bigvee_{(D,E)\in\mathcal{R}_i} \bigcirc\hat{E}$	*lemma 7.7 on (4)*
(6)	$\vdash \hat{D} \Rightarrow \bigcirc \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}$	*lemma 7.6 on (3), (5)*
(7)	$\vdash \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \Rightarrow \bigcirc \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}$	*lemma 7.11 on (6)*
(8)	$\vdash \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \Rightarrow \neg\alpha$	*lemma 7.13 on (i)*
(9)	$\vdash \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \Rightarrow \Diamond\alpha$	*lemma 7.13 on (ii)*
(10)	$\vdash \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \Rightarrow \Box\neg\alpha$	ind *on (7), (8)*
(11)	$\vdash \neg \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \vee \Box\neg\alpha$	*def. of* $\Rightarrow$ *on (10)*
(12)	$\vdash \neg \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}, \Box\neg\alpha$	$\vee$ *on (10)*
(13)	$\vdash \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \Rightarrow \Box\Diamond\alpha$	ind *on (7), (9)*
(14)	$\vdash \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \Rightarrow \Diamond\alpha$	*lemma 7.12 on (13)*
(15)	$\vdash \neg \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B} \vee \Diamond\alpha$	*def. of* $\Rightarrow$ *on (14)*
(16)	$\vdash \neg \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}, \Diamond\alpha$	$\vee$ *on (15)*
(17)	$\vdash \neg \bigvee_{(C,B)\in\mathcal{R}_i^*} \hat{B}$	a-cut *on (12), (16)*
(18)	$\vdash \neg \bigvee_{C\in\mathcal{C}} \hat{C}$	$\mathcal{C}$ *is strongly connected*

The rule a-cut *is allowed in step (17) because if $\Diamond\alpha \in \mathsf{Cl}(\varphi)$, then $\neg\Diamond\alpha = \Box\neg\alpha \in \mathsf{Cl}(\varphi)$ and thus $\Diamond\alpha, \Box\neg\alpha \in \mathsf{FL}(\varphi)$.*
Now we have shown that all atoms in the strongly connected subgraph that is eliminated in the passage from step i to step $i+1$ are provably false. For **(I)** *we have:*

(19)	$\vdash \varphi \Rightarrow \bigvee_{\varphi\in A\in\mathcal{A}_i} \hat{A}$	*IH*
(20)	$\vdash \neg\varphi \vee \bigvee_{\varphi\in A\in\mathcal{A}_i} \hat{A}$	*def. of $\Rightarrow$ on (19)*
(21)	$\vdash \neg\varphi, \bigvee_{\varphi\in A\in\mathcal{A}_i} \hat{A}$	*$\vee$ on (20)*
(22)	$\vdash \neg\varphi, \bigvee_{\varphi\in A\in\mathcal{A}_{i+1}} \hat{A} \vee \bigvee_{\varphi\in A\in\mathcal{A}_i\setminus\mathcal{A}_{i+1}} \hat{A}$	*disjuntion on (22)*
(23)	$\vdash \neg\varphi, \bigvee_{\varphi\in A\in\mathcal{A}_{i+1}} \hat{A}, \bigvee_{\varphi\in A\in\mathcal{A}_i\setminus\mathcal{A}_{i+1}} \hat{A}$	*$\vee$ on (22)*
(24)	$\vdash \varphi \Rightarrow \bigvee_{\varphi\in A\in\mathcal{A}_{i+1}} \hat{A}$	a-cut *on (18), (23)*

The use of a-cut *in step (24) is allowed because we cut elements of* $\mathsf{Cl}(\varphi)$*.*
For **(II)** *we have:*

(25)	$\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{(A,B)\in\mathcal{R}_i} \hat{B}$	*IH*
(26)	$\vdash \hat{A} \Rightarrow (\bigcirc \bigvee_{(A,B)\in\mathcal{R}_{i+1}} \hat{B} \vee \bigcirc \bigvee_{(A,B)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \hat{B})$	*disj. of (25)*
(27)	$\vdash \neg\hat{A} \vee \bigcirc \bigvee_{(A,B)\in\mathcal{R}_{i+1}} \hat{B} \vee \bigcirc \bigvee_{(A,B)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \hat{B}$	*def. of $\Rightarrow$ on (26)*
(28)	$\vdash \neg\hat{A}, \bigcirc \bigvee_{(A,B)\in\mathcal{R}_{i+1}} \hat{B}, \bigcirc \bigvee_{(A,B)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \hat{B}$	*$\vee$ on (27)*
(29)	$\vdash \neg \bigvee_{(A,B)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \hat{B}$	*lemma 7.4 on (18)*
(30)	$\vdash \neg\hat{A}, \bigcirc\neg \bigvee_{(A,B)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \hat{B}$	*$\bigcirc$ on (29)*
(31)	$\vdash \neg\hat{A}, \bigcirc \bigvee_{(A,B)\in\mathcal{R}_{i+1}} \hat{B}$	a-cut *on (28), (30)*
(32)	$\vdash \neg\hat{A} \vee \bigcirc \bigvee_{(A,B)\in\mathcal{R}_{i+1}} \hat{B}$	*$\vee$ on (31)*
(33)	$\vdash \hat{A} \Rightarrow \bigcirc \bigvee_{(A,B)\in\mathcal{R}_{i+1}} \hat{B}$	*def. of $\Rightarrow$ on (32)*

In step (29) we use the result (18). which states that $\vdash \neg\bigvee_{C\in\mathcal{C}} \hat{C} = \bigwedge_{C\in\mathcal{C}} \neg\hat{C}$. Thus, we take from this set the elements $\hat{B}$ such that $(A,B) in \mathcal{R}_{i+1}\setminus\mathcal{R}_i$ using lemma 7.4. The use of a-cut *in step is justified because the cut-formulæ are of the form $\bigcirc\bigvee\hat{B}$ and $\bigcirc\neg\bigvee\hat{B}$. By lemma 7.7 we may push the next-operators $\bigcirc$ inward and thus by lemma 7.6 we cut formulæ of the form $\bigcirc\alpha$, $\bigcirc\neg\alpha$ with $\alpha, \neg\alpha \in \mathsf{Cl}(\varphi)$. Thus, the cut-formulæ belong to* FL*(φ).*
For **(III)***, if we call $R = \{(A,C) \mid (A,C) \notin \mathcal{R}_{i+1}\}$, we have by induction hypothesis that*

$$R = \{(A,C) \mid (A,C) \notin \mathcal{R}_i\} \cup \{(A,C) \mid (A,C) \in \mathcal{R}_i\setminus\mathcal{R}_{i+1}\} \; (*)$$

Thus:

(34)	$\vdash \hat{A} \Rightarrow \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_i} \neg\hat{C}$	**(III)** *IH*
(35)	$\vdash \bigwedge_{C\in\mathcal{C}} \neg\hat{C}$	*def. of neg. on (18)*
(36)	$\vdash \neg\hat{A}, \bigcirc \bigwedge_{C\in\mathcal{C}} \neg\hat{C}$	$\bigcirc$ *on (35)*
(37)	$\vdash \neg\hat{A}, \bigcirc \bigwedge_{(A,C)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \neg\hat{C}$	*see below*
(38)	$\vdash \hat{A} \Rightarrow (\bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_i} \neg\hat{C} \wedge \bigcirc \bigwedge_{(A,C)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \neg\hat{C})$	$\wedge$ *on (34), (37)*
(39)	$\vdash \hat{A} \Rightarrow \bigcirc(\bigwedge_{(A,C)\notin\mathcal{R}_i} \neg\hat{C} \wedge \bigwedge_{(A,C)\in\mathcal{R}_i\setminus\mathcal{R}_{i+1}} \neg\hat{C})$	*lemma 7.14 on (38)*
(40)	$\vdash \neg\hat{A}, \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_{i+1}} \neg\hat{C}$	*by (*) above*
(41)	$\vdash \neg\hat{A} \vee \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_{i+1}} \neg\hat{C}$	$\vee$ *on (40)*
(42)	$\vdash \hat{A} \Rightarrow \bigcirc \bigwedge_{(A,C)\notin\mathcal{R}_{i+1}} \neg\hat{C}$	*def. of* $\Rightarrow$ *on (41)*

Step (34) is justified because all atoms $C \in \mathcal{C}$ are eliminated in the passage from step i to $i+1$. Hence, for any atom A, $C \in \mathcal{C}$ implies that $(A, C) \in \mathcal{R}_i\setminus\mathcal{R}_{i+1}$.

Theorem 6.8 (Completeness of S_0)**.** *If a formula φ is unsatisfiable, then $S_0 \vdash \neg\varphi$.*

Proof *If φ is unsatisfiable, then the algorithm ends with an empty set $\mathcal{A}_n$ after n steps. We have thus:*

(1)	$\vdash \varphi \Rightarrow \bigvee_{\varphi\in A\in\mathcal{A}_n} \hat{A}$	*theorem 6.7*
(2)	$\vdash \neg\varphi \vee \bigvee_{\varphi\in A\in\mathcal{A}_n} \hat{A}$	*def. of* $\Rightarrow$ *on (1)*
(3)	$\vdash \neg\varphi, \bigvee_{\varphi\in A\in\mathcal{A}_n} \hat{A}$	$\vee$ *on (2)*
(4)	$\vdash \neg\varphi, \bigvee \emptyset$	*hyp. on (3)*
(5)	$\vdash \neg\varphi, \bot$	*def. of* $\bigvee$ *on (4)*
(6)	$\vdash \neg\varphi$	*lemma 7.8 on (6)*

As a final observation, cuts are also used in auxiliary lemmas 7.4 and 7.5. Lemma 7.4 is used in theorem 6.7. It is used on the formula $\neg \bigvee_{C\in\mathcal{C}} \hat{C}$, where C is an atom. Thus the cut which is applied in lemma 7.4 is on formulæ belonging to FL(φ). Lemma 7.5 is used in lemma 6.1 on formulæ belonging to Cl(φ) and therefore it is acceptable. Finally, lemma 7.4 is used to prove lemma 7.5. In this case also we are using a cut-formula of Cl(φ).

7 Auxiliary Lemmas

Lemma 7.1. *The rule* $\vee$ *is invertible.*

Proof *The proof amounts to showing the admissibility of the rule* $\vee$ *inverted, namely*

$$\vee\text{-inv}\ \frac{\Gamma, \alpha \vee \beta}{\Gamma, \alpha, \beta}$$

The base case is the id *rule. In this case, since the disjunction appears as a result of weakening, it suffices to change this formula. The same holds for the rules* $\bigcirc$*,* ind*, and* $\Box\Diamond$*. The rule* $\vee$-inv *is passive in the cases of the rules* $\wedge$*,* $\Box$*,* $\Diamond$*, and* a-cut *and may thus be safely pushed upwards. In the case of the rule* $\vee$*, it is enough to retain the premise.*

Lemma 7.2. *For any formula* α *we have that* $\vdash \Gamma, \alpha, \neg\alpha$*.*

Proof *Induction on the structure of* α*.*
Base case*. If* $\alpha = p$*, then this is rule* id*.*
Induction step*. We consider the different cases.*

- $\alpha = \neg\beta$*. By induction hypothesis we have that* $\vdash \Gamma, \beta, \neg\beta$*. The result follows from the equivalence* $\alpha \equiv \neg\neg\alpha$*.*

- $\alpha = \beta \vee \delta$*. Then we have:*

(1)	$\vdash \Gamma, \beta, \delta, \neg\beta$	*IH*
(2)	$\vdash \Gamma, \beta, \delta, \neg\delta$	*IH*
(3)	$\vdash \Gamma, \beta, \delta, \neg\beta \wedge \neg\delta$	$\wedge$ *on (1), (2)*
(4)	$\vdash \Gamma, \beta \vee \delta, \neg\beta \wedge \neg\delta$	$\vee$ *on (3)*
(5)	$\vdash \Gamma, \beta \vee \delta, \neg(\beta \vee \delta)$	*def. of neg. on (4)*

- $\alpha = \beta \wedge \delta$*. Then we have:*

(1)	$\vdash \Gamma, \neg\beta, \neg\delta, \beta$	*IH*
(2)	$\vdash \Gamma, \neg\beta, \neg\delta, \delta$	*IH*
(3)	$\vdash \Gamma, \neg\beta, \neg\delta, \beta \wedge \delta$	$\wedge$ *on (1), (2)*
(4)	$\vdash \Gamma, \neg\beta \vee \neg\delta, \beta \wedge \delta$	$\vee$ *on (3)*
(5)	$\vdash \Gamma, \neg(\beta \wedge \delta), \beta \wedge \delta$	*def. of neg. on (4)*

- $\alpha = \bigcirc\beta$. *Then we have:*

$$
\begin{array}{lll}
(1) & \vdash \beta, \neg\beta & \textit{IH} \\
(2) & \vdash \Gamma, \bigcirc\beta, \bigcirc\neg\beta & \bigcirc \textit{ on (1)} \\
(3) & \vdash \Gamma, \bigcirc\beta, \neg \bigcirc \beta & \textit{def. of neg. on (2)}
\end{array}
$$

- $\alpha = \Diamond\beta$. *Then we have:*

$$
\begin{array}{lll}
(1) & \vdash \neg\beta, \beta & \textit{IH} \\
(2) & \vdash \Gamma, \Diamond\beta, \Box\neg\beta & \Box\Diamond \textit{ on (1)} \\
(3) & \vdash \Gamma, \Diamond\beta, \neg\Diamond\neg\neg\beta & \textit{def. of neg. on (2)} \\
(4) & \vdash \Gamma, \Diamond\beta, \neg\Diamond\beta & \textit{def. of neg. on (3)}
\end{array}
$$

- $\alpha = \Box\beta$. *Then we have:*

$$
\begin{array}{lll}
(1) & \vdash \neg\beta, \beta & \textit{IH} \\
(2) & \vdash \Gamma, \Box\beta, \Diamond\neg\beta & \Box\Diamond \textit{ on (1)} \\
(3) & \vdash \Gamma, \Box\beta, \neg\Box\neg\neg\beta & \textit{def. neg. on (2)} \\
(4) & \vdash \Gamma, \Box\beta, \neg\Box\beta & \textit{def. neg. on (3)}
\end{array}
$$

Lemma 7.3. *Weakening is admissible in $S_0(\varphi)$.*

Proof *Admissibility proofs will be carried out by induction on the length of the proof. The base case is the proof that weakening is not necessary in the case of the* id *rule. The induction step is the proof that for each rule, weakening may be pushed upwards (i.e., nearer the axioms.)*
In the case of the rules id*,* $\bigcirc$*,* ind*, and* $\Box\Diamond$*, there is weakening involved in the rules, so that it may be embedded. For instance, the derivation:*

$$
\begin{array}{lll}
(1) & \vdash \Gamma & \textit{hyp.} \\
(2) & \vdash \bigcirc\Gamma, \Sigma & \bigcirc \textit{ on (1)} \\
(3) & \vdash \bigcirc\Gamma, \Sigma, \alpha & \textit{weakening on (2)}
\end{array}
$$

may be replaced by:

$$
\begin{array}{lll}
(1) & \vdash \Gamma & \textit{hyp.} \\
(2) & \vdash \bigcirc\Gamma, \Sigma, \alpha & \bigcirc \textit{ on (1)}
\end{array}
$$

In the case of all other rules, weakening is passive and may be safely pushed upwards. For instance, the derivation:

(1)	$\vdash \Gamma, \alpha$	*hyp.*
(2)	$\vdash \Gamma, \beta$	*hyp.*
(3)	$\vdash \Gamma, \alpha \wedge \beta$	*$\wedge$ on (1), (2)*
(4)	$\vdash \Gamma, \alpha \wedge \beta, \varphi$	*weakening on (3)*

may be replaced by

(1)	$\vdash \Gamma, \alpha$	*hyp.*
(2)	$\vdash \Gamma, \beta$	*hyp.*
(3)	$\vdash \Gamma, \alpha, \varphi$	*weakening on (1)*
(4)	$\vdash \Gamma, \beta, \varphi$	*weakening on (2)*
(5)	$\vdash \Gamma, \alpha \wedge \beta, \varphi$	*$\wedge$ on (3), (4)*

As a corollary, we may note that the formulation of the a-cut *is equivalent to the (slightly more general) following one:*

$$\textsf{a-cut'}\ \frac{\Gamma, \Sigma_1, \alpha \quad \Gamma, \Sigma_2, \neg\alpha}{\Gamma, \Sigma_1, \Sigma_2}$$

This may be easily proved as follows

(1)	$\vdash \Gamma, \Sigma_1, \alpha$	*right hyp.*
(2)	$\vdash \Gamma, \Sigma_1, \Sigma_2, \alpha$	*weakening on (1)*
(3)	$\vdash \Sigma_2, \neg\alpha$	*left hyp.*
(4)	$\vdash \Gamma, \Sigma_1, \Sigma_2, \neg\alpha$	*weakening on (3)*
(5)	$\vdash \Gamma, \Sigma_1, \Sigma_2$	a-cut *on (2), (4)*

Lemma 7.4. *The rule*

$$\textsf{R}\ \frac{\Gamma, \alpha \wedge \beta}{\Gamma, \alpha}$$

is admissible.

Proof *The rule* R *is passive in the case of the rules $\vee$, $\Box$, and $\Diamond$. Therefore it can be safely pushed upwards in these cases. The same holds for the rule $\wedge$ when* R *is passive. In the latter case, if the rule* R *is not passive, then it suffices to retain the corresponding premise of the rule.*
In the case of rules id*, $\bigcirc$,* ind*, and $\Box\Diamond$, the rule is applicable on a formula that appears as a consequence of weakening. Therefore, it suffices to modify this formula. For instance, the derivation*

(1)	$\vdash \Gamma, \alpha$	*hyp.*
(2)	$\vdash \Diamond\Gamma, \Box\alpha, \beta \wedge \delta$	$\Box\Diamond$ *on (1)*
(3)	$\vdash \Diamond\Gamma, \Box\alpha, \beta$	*R on (2)*

may be replaced by

(1)	$\vdash \Gamma, \alpha$	*hyp.*
(2)	$\vdash \Diamond\Gamma, \Box\alpha, \beta$	$\Box\Diamond$ *on (1)*

The case of the rule a-cut *deserves a closer look. Assume we have:*

(1)	$\vdash \Gamma, \alpha \wedge \beta, \delta$	*hyp.*
(2)	$\vdash \Gamma, \alpha \wedge \beta, \neg\delta$	*hyp.*
(3)	$\vdash \Gamma, \alpha \wedge \beta$	a-cut *on (1)*
(4)	$\vdash \Gamma, \alpha$	R *on (3)*

Since $\delta \in \mathsf{FL}(\varphi)$*, nothing changes when we push the rule upwards (the* a-cut *rule is always applicable.)*

Lemma 7.5. *If* $\vdash (\alpha \vee \beta) \wedge \delta$*, then* $\vdash \delta \wedge \alpha, \delta \wedge \beta$*.*

Proof

(1)	$\vdash (\alpha \vee \beta) \wedge \delta$	*hyp.*
(2)	$\vdash \alpha \vee \beta$	*lemma 7.4 on (1)*
(3)	$\vdash \alpha, \beta$	$\vee$ *on (2)*
(4)	$\vdash \delta$	*lemma 7.4 on (1)*
(5)	$\vdash \delta, \alpha$	*lemma 7.3 on (4)*
(6)	$\vdash \alpha, \delta \wedge \beta$	$\wedge$ *on (3), (5)*
(7)	$\vdash \delta, \delta \wedge \beta$	*lemma 7.3 on (4)*
(8)	$\vdash \delta \wedge \alpha, \delta \wedge \beta$	$\wedge$ *on (6), (7)*

Lemma 7.6. *Let* $\hat{\Psi}_i$*,* $1 \leq i \leq p$*, be a collection of conjunctions of formulæ belonging to* $\mathsf{FL}(\varphi)$*, i.e.,* $\hat{\Psi}_i = \psi_{i,1} \wedge \psi_{i,2} \wedge \ldots \wedge \psi_{i,m_i}$ *with* $\psi_{i,j} \in \mathsf{FL}(\varphi)$*. Let further*

$$\vdash \Gamma, \bigvee_{1 \leq i \leq n} \hat{\Psi}_i \qquad (i)$$
$$\vdash \Gamma, \bigwedge_{1 \leq j \leq p} \neg\hat{\Psi}_j \qquad (ii)$$

for some p*,* n *such that* $p \leq n$*. Then* $\vdash \Gamma, \bigvee_{p < h \leq n} \hat{\Psi}_h$*.*

Proof *We begin by eliminating $\hat{\Psi}_1 = \psi_1 \wedge \ldots \wedge \psi_q$. We have:*

(1)	$\vdash \Gamma, \hat{\Psi}_1 \vee \ldots \vee \hat{\Psi}_n$	*hyp. (i)*
(2)	$\vdash \Gamma, \hat{\Psi}_1, \ldots, \hat{\Psi}_n$	$\vee$ *on (1)*
(3)	$\vdash \Gamma, (\psi_1 \wedge \ldots \wedge \psi_q), \hat{\Psi}_2, \ldots, \hat{\Psi}_n$	*def of* $\hat{\Psi}_1$ *on (2)*
(4.1)	$\vdash \Gamma, \psi_1, \hat{\Psi}_2, \ldots, \hat{\Psi}_n$	*lemma 7.4 on (1)*
(4.2)	$\vdash \Gamma, \psi_2, \hat{\Psi}_2, \ldots, \hat{\Psi}_n$	*lemma 7.4 on (1)*
	. . . .	
(4.q)	$\vdash \Gamma, \psi_q, \hat{\Psi}_2, \ldots, \hat{\Psi}_n$	*lemma 7.4 on (1)*
(5)	$\vdash \Gamma, \neg\hat{\Psi}_1 \wedge \ldots \wedge \neg\hat{\Psi}_p$	*hyp. (ii)*
(6)	$\vdash \Gamma, \neg\hat{\Psi}_1$	*lemma 7.4 on (5)*
(7)	$\vdash \Gamma, \neg(\psi_1 \wedge \ldots \wedge \psi_q)$	*def. of* $\hat{\Psi}_1$ *on (6)*
(8)	$\vdash \Gamma, \neg\psi_1 \vee \ldots \vee \neg\psi_q$	*def. of neg. on (7)*
(9)	$\vdash \Gamma, \neg\psi_1, \ldots, \neg\psi_q$	$\vee$ *on (8)*
(10.1)	$\vdash \Gamma, \hat{\Psi}_2, \ldots, \hat{\Psi}_n, \neg\psi_2, \ldots \neg\psi_q$	a-cut' *on (4.1), (9)*
(10.2)	$\vdash \Gamma, \hat{\Psi}_2, \ldots, \hat{\Psi}_n, \neg\psi_3, \ldots, \neg\psi_q$	a-cut' *on (4.2), (10.1)*
	. . . .	
(10.q)	$\vdash \Gamma, \hat{\Psi}_2, \ldots, \hat{\Psi}_n$	a-cut' *on (4.q),(10.q-1)*

In steps (10.1) to (10.q) the rule a-cut' *is the version of* a-cut *mentioned at the end of lemma 7.3. In the same way, we proceed to eliminate* $\hat{\Psi}_2, \ldots, \hat{\Psi}_p$. *This yields finally*

$$\vdash \Gamma, \hat{\Psi}_{p+1}, \ldots, \hat{\Psi}_n$$

Observe that in all cases we have used cut-formulæ belonging to $\mathsf{FL}(\varphi)$.

Lemma 7.7. *the rules*

$$\mathsf{R}_1 \ \frac{\Gamma, \bigcirc(\alpha \wedge \beta)}{\Gamma, \bigcirc\alpha \wedge \bigcirc\beta}$$

and

$$\mathsf{R}_2 \ \frac{\Gamma, \bigcirc(\alpha \vee \beta)}{\Gamma, \bigcirc\alpha \vee \bigcirc\beta}$$

are admissible.

Proof *In the case of rules* id, ind*, and* $\Box\Diamond$*, the premises appear as a consequence of weakening. Thus, it is enough to modify this formula. For instance, the derivation*

(1)	$\vdash \Gamma, p, \neg p, \bigcirc(\alpha \vee \beta)$	id
(2)	$\vdash \Gamma, p, \neg p, \bigcirc\alpha \vee \bigcirc\beta$	R_2 *on (1)*

may be replaced by

(1)	$\vdash \Gamma, p, \neg p, \bigcirc\alpha \vee \bigcirc\beta$	id

In the cases of the rules $\wedge$, $\vee$, $\Box$, $\Diamond$, and a-cut*, the rules are passive and may be safely pushed upwards. For instance, the derivation*

(1)	$\vdash \Gamma, \varphi, \psi, \bigcirc(\alpha \vee \beta)$	*hyp.*
(2)	$\vdash \Gamma, \varphi \vee \psi, \bigcirc(\alpha \vee \beta)$	$\vee$ *on (1)*
(3)	$\vdash \Gamma, \varphi \vee \psi, \bigcirc\alpha \vee \bigcirc\beta$	R_2 *on (2)*

may be replaced by

(1)	$\vdash \Gamma, \varphi, \psi, \bigcirc(\alpha \vee \beta)$	*hyp.*
(2)	$\vdash \Gamma, \varphi, \psi, \bigcirc\alpha \vee \bigcirc\beta$	R_2 *on (1)*
(3)	$\vdash \Gamma, \varphi \vee \psi, \bigcirc\alpha \vee \bigcirc\beta$	$\vee$ *on (2)*

It remains to consider the rule $\bigcirc$. The derivation

(1)	$\vdash \alpha \wedge \beta$	*hyp.*
(2)	$\vdash \Gamma, \bigcirc(\alpha \wedge \beta)$	$\bigcirc$ *on (1)*
(3)	$\vdash \Gamma, \bigcirc\alpha \wedge \bigcirc\beta$	R_1 *on (2)*

may be replaced by

(1)	$\vdash \alpha \wedge \beta$	*hyp.*
(2)	$\vdash \alpha$	*lemma 7.4 on (1)*
(3)	$\vdash \Gamma, \bigcirc\alpha$	$\bigcirc$ *on (2)*
(4)	$\vdash \beta$	*lemma 7.4 on (1)*
(5)	$\vdash \Gamma, \bigcirc\beta$	$\bigcirc$ *on (4)*
(6)	$\vdash \Gamma, \bigcirc\alpha \wedge \bigcirc\beta$	$\wedge$ *on (5)*

and the derivation

(1)	$\vdash \alpha \vee \beta$	*hyp.*
(2)	$\vdash \Gamma, \bigcirc(\alpha \vee \beta)$	$\bigcirc$ *on (1)*
(3)	$\vdash \Gamma, \bigcirc\alpha \vee \bigcirc\beta$	R_2 *on (2)*

may be replaced by

(1)	$\vdash \alpha \vee \beta$	*hyp.*
(2)	$\vdash \alpha, \beta$	$\vee$ *on (1)*
(3)	$\vdash \Gamma, \bigcirc\alpha, \bigcirc\beta$	$\bigcirc$ *on (2)*
(4)	$\vdash \Gamma, \bigcirc\alpha \vee \bigcirc\beta$	$\vee$ *on (3)*

Lemma 7.8. *If* $\vdash \Gamma, \bot$ *then* $\vdash \Gamma$.

Proof

(1)	$\vdash \Gamma, \bot$	*hyp.*
(2)	$\vdash \Gamma, p \wedge \neg p$	*def. of* $\bot$ *on (1)*
(3)	$\vdash \Gamma, p$	*lemma 7.4 on (2)*
(4)	$\vdash \Gamma, \neg p$	*lemma 7.4 on (2)*
(5)	$\vdash \Gamma$	a-cut *on (3), (4)*

In step (2) we choose some $p \in \mathsf{Suf}(\varphi)$ *to be able to apply* a-cut *in step (5). This is always possible, unless* φ *is empty.*

Lemma 7.9. $\vdash \alpha \Rightarrow (\alpha \vee \beta)$.

Proof

(1)	$\vdash \neg\alpha, \alpha, \beta$	*lemma 7.2*
(2)	$\vdash \neg\alpha \vee \alpha \vee \beta$	$\vee$ *on (1)*
(2)	$\vdash \alpha \Rightarrow (\alpha \vee \beta)$	*def. of* $\Rightarrow$ *on (2)*

Lemma 7.10. *If* $\vdash \alpha \Rightarrow \beta$, *then* $\vdash \bigcirc\alpha \Rightarrow \bigcirc\beta$.

Proof

(1)	$\vdash \alpha \Rightarrow \beta$	*hyp.*
(2)	$\vdash \neg\alpha \vee \beta$	*def. of* $\Rightarrow$ *on (1)*
(3)	$\vdash \neg\alpha, \beta$	$\vee$ *on (2)*
(4)	$\vdash \bigcirc\neg\alpha, \bigcirc\beta$	$\bigcirc$ *on (2)*
(5)	$\vdash \neg \bigcirc \alpha, \bigcirc\beta$	*def. of neg. on (4)*
(6)	$\vdash \neg \bigcirc \alpha \vee \bigcirc\beta$	$\vee$ *on (5)*
(7)	$\vdash \bigcirc\alpha \Rightarrow \bigcirc\beta$	*def. of* $\Rightarrow$ *on (6)*

Lemma 7.11. *If* $\vdash \alpha_1 \Rightarrow \beta$ *and* $\vdash \alpha_2 \Rightarrow \beta$, *then* $\vdash (\alpha_1 \vee \alpha_2) \Rightarrow \beta$.

Proof

(1)	$\vdash \alpha_1 \Rightarrow \beta$	*hyp.*
(2)	$\vdash \neg\alpha_1 \vee \beta$	*def. of* $\Rightarrow$ *on (1)*
(3)	$\vdash \neg\alpha_1, \beta$	$\vee$ *on (2)*
(4)	$\vdash \alpha_2 \Rightarrow \beta$	*hyp.*
(5)	$\vdash \neg\alpha_1 \vee \beta$	*def. of* $\Rightarrow$ *on (4)*
(6)	$\vdash \neg\alpha_1, \beta$	$\vee$ *on (5)*
(7)	$\vdash \neg\alpha_1 \wedge \neg\alpha_2, \beta$	$\wedge$ *on (3) and (6)*
(8)	$\vdash \neg(\alpha_1 \vee \neg\alpha_2), \beta$	*def. of neg. on (7)*
(9)	$\vdash \neg(\alpha_1 \vee \neg\alpha_2) \vee \beta$	$\vee$ *on (8)*
(10)	$\vdash (\alpha_1 \vee \alpha_2) \Rightarrow \beta$	*def. of* $\Rightarrow$ *on (9)*

Lemma 7.12. *The rule*

$$\mathsf{R}\ \frac{\Gamma, \Box\alpha}{\Gamma, \alpha}$$

is admissible.

Proof *The rule* R *is passive in the case of the rules* $\vee$*,* $\wedge$*,* $\Diamond$*, and* a-cut*. Therefore it can be safely pushed upwards in these cases. The formula* $\Box\alpha$ *appears as a consequence of weakening in rules* id *and* $\bigcirc$ *and then it suffices to change the weakening formula. The same holds when the formula* $\Box\alpha$ *appears as a consequence of weakening in the* ind *rule.*
It remains to consider the rules $\Box$*,* ind*, and* $\Box\Diamond$*.*
In the case of rule $\Box$ *we can simply use the left premise. In the case of rule* ind*, we can apply weakening to the right premise. Now we consider the rule* $\Box\Diamond$*. The derivation*

(1)	$\vdash \Gamma, \varphi$	*hyp.*
(2)	$\vdash \Diamond\Gamma, \Box\varphi, \Sigma$	$\Box\Diamond$ *on (1)*
(3)	$\vdash \Diamond\Gamma, \varphi, \Sigma$	*R on (2)*

may be replaced by

(1)	$\vdash \Gamma, \varphi$	*hyp.*
(2)	$\vdash \Gamma, \bigcirc\Diamond\Gamma, \varphi, \Sigma$	*lemma 7.3 on (1)*
(3)	$\vdash \Diamond\Gamma, \varphi, \Sigma$	$\Diamond$ *on (2)*

Lemma 7.13. $\vdash (\alpha \wedge \beta) \Rightarrow \alpha$*.*

Proof

(1)	$\vdash \neg\alpha, \neg\beta, \alpha$	*lemma 7.2*
(2)	$\vdash \neg\alpha \vee \neg\beta \vee \alpha$	*$\vee$ on (1)*
(3)	$\vdash \neg(\alpha \wedge \beta) \vee \alpha$	*def. of neg. on (2)*
(4)	$\vdash (\alpha \wedge \beta) \Rightarrow \alpha$	*def. of $\Rightarrow$ on (3)*

Lemma 7.14. *Let $\hat{\Psi}_i$, $1 \leq i \leq p$, and $\hat{\Psi}_j$, $p < j \leq n$, be two collections of conjunctions of formulæ belonging to $\mathsf{Cl}(\varphi)$ and let $\vdash \Gamma, \bigcirc \bigwedge_i \neg\hat{\Psi}_i \wedge \bigcirc \bigwedge_j \neg\hat{\Psi}_j$. Then $\vdash \Gamma, \bigcirc(\bigwedge_i \neg\hat{\Psi}_i \wedge \bigwedge_j \neg\hat{\Psi}_j)$.*

Proof

(1)	$\vdash \bigvee_i \hat{\Psi}_i,\ \bigvee_j \hat{\Psi}_j,\ \neg \bigvee_i \hat{\Psi}_i$	*lemma 7.2*
(2)	$\vdash \bigvee_i \hat{\Psi}_i,\ \bigvee_j \hat{\Psi}_j,\ \bigwedge_i \neg\hat{\Psi}_i$	*def. of neg. on (1)*
(3)	$\vdash \bigvee_i \hat{\Psi}_i,\ \bigvee_j \hat{\Psi}_j,\ \neg \bigvee_j \hat{\Psi}_j$	*lemma 7.2*
(4)	$\vdash \bigvee_i \hat{\Psi}_i,\ \bigvee_j \hat{\Psi}_j,\ \bigwedge_j \neg\hat{\Psi}_j$	*def. of neg. on (3)*
(5)	$\vdash \bigvee_i \hat{\Psi}_i,\ \bigvee_j \hat{\Psi}_j,\ \bigwedge_i \hat{\Psi}_i \wedge \bigwedge_j \hat{\Psi}_j$	*$\wedge$ on (2), (4)*
(6)	$\vdash \Gamma,\ \bigcirc \bigvee_i \hat{\Psi}_i,\ \bigcirc \bigvee_j \hat{\Psi}_j,\ \bigcirc(\bigwedge_i \hat{\Psi}_i \wedge \bigwedge_j \hat{\Psi}_j)$	*$\bigcirc$ on (5)*
(7)	$\vdash \Gamma,\ \bigcirc \bigwedge_i \neg\hat{\Psi}_i \wedge \bigcirc \bigwedge_j \neg\hat{\Psi}_j$	*hyp.*

Now let us consider the expressions $\bigcirc \bigvee_i \hat{\Psi}_i$ and $\bigcirc \bigwedge_i \neg\hat{\Psi}_i$. By lemma 7.7 we may push the next-operator $\bigcirc$ inside the expressions. Since $\hat{\Psi}_i$ is a conjunction of formulæ belonging to $\mathsf{Cl}(\varphi)$, we get a conjunction of formulæ belonging to $\mathsf{FL}(\varphi)$ when we prefix them with $\bigcirc$. We denote this by $\bigvee_i \hat{\Psi}'_i$ and $\bigwedge_i \neg\hat{\Psi}'_i$ respectively. Thus we have:

(6')	$\vdash \Gamma,\ \bigvee_i \hat{\Psi}'_i,\ \bigvee_j \hat{\Psi}'_j,\ \bigcirc(\bigwedge_i \hat{\Psi}_i \wedge \bigwedge_j \hat{\Psi}_j)$	*(6) rewritten*
(7')	$\vdash \Gamma,\ \bigwedge_i \neg\hat{\Psi}'_i \wedge \bigwedge_j \neg\hat{\Psi}'_j$	*(7) rewritten*
(8)	$\vdash \Gamma,\ \bigwedge_i \neg\hat{\Psi}'_i$	*lemma 7.4 on (7')*
(9)	$\vdash \Gamma,\ \bigvee_j \hat{\Psi}'_j,\ \bigcirc(\bigwedge_i \hat{\Psi}_i \wedge \bigwedge_j \hat{\Psi}_j)$	*lemma 7.6 on (6'), (8)*
(10)	$\vdash \Gamma,\ \bigwedge_j \neg\hat{\Psi}'_j$	*lemma 7.4 on (7')*
(11)	$\vdash \Gamma,\ \bigcirc(\bigwedge_i \hat{\Psi}_i \wedge \bigwedge_j \hat{\Psi}_j)$	*lemma 7.6 on (9), (10)*

8 Conclusions and Future Work

In this paper we have presented a calculus which uses an analytical cut rule, in which the cut-formula is restricted to an element of the Fischer-Ladner closure of Γ, where Γ is the sequent whose validity is to be proved. The search for a cut formula is thus limited to the power set of $\mathsf{FL}(\Gamma)$, which is linear on the size of Γ_0. Hence the complexity of the problem is nonetheless exponential.

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