A Sequent System for Sets of Clauses

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October 2006

Abstract

Sets of clauses are often used to represent the knowledge or beliefs of an agent. Usually sets of clauses represent incomplete information, since the agent does not have a complete "picture of the world." Thus, there is usually not a single model but a set of models that satisfy the set of clauses representing the worlds that the agent considers possible. We define the invariant of a propositional set of clauses as the set of belong to all models and present a sequent system that may be used to construct such a set. We prove that the system is sound (all atoms that can be inferred with it are in the invariant), complete (all atoms in the invariant may be inferred with the sequent system) and that the computation ends after a finite number of steps.

Klassification: F.4.1 Mathematical Logic - Proof Theory
General Terms: Theory
Additional Key Words: Multi-Agent Systems, Logic Programming

1 Introduction and Basic Definitions

Sets of clauses are often used to represent the knowledge or beliefs of an agent. Usually sets of clauses represent incomplete information, since the agent does not have a complete "picture of the world." Thus, there is usually not a single model but a set of models that satisfy the set of clauses representing the worlds that the agent considers possible. We define the invariant of a propositional set of clauses as the set of belong to all models and present a sequent system that may be used to construct such a set. We prove that the system is sound (all atoms that can be inferred with it are in the invariant), complete (all atoms in the invariant may be inferred with the sequent system) and that the computation ends after a finite number of steps. We will start with a set of propositional symbols Π. An *atom* will be either a symbol of Π or its negation. Thus there are *positive*, *negative* and *complementary* atoms. The set of all atoms that can be constructed with a set of propositions Π is denoted by $\mathsf{AT}(\Pi)$. Usually we will denote atoms by lowercase sans-serif letters (p) without specifying whether they positive or negative. We will explicitly use p or $\overline{\mathsf{p}}$ when we want to explicitly denote positive or negative atoms. As usual, $\overline{\overline{\mathsf{p}}} = \mathsf{p}$.

2 Syntax and Semantics

The syntax of the language is given by the following grammar:

$$\begin{aligned} \mathsf{sequent} &::= \varepsilon \mid \mathsf{atom}[, \mathsf{atom}] \\ \mathsf{clause} &::= \mathsf{sequent} \rightarrow \mathsf{atom} \end{aligned}$$

We will use Γ to denote sequents. A *knowledge database* (KB), denoted by Δ, is a finite set of clauses. If the set Π contains the propositions that occur in Δ, we will say that Δ is *based* on Π. The notation $\mathcal{C}(\Delta)$ will be used to denote the list of all propositions p such that both p and $\overline{\mathsf{p}}$ occur in Δ. As usual, the intended meaning of a clause is that if the antecedent is true, so is the consequent.

A *world* of Π is a maximal consistent subset of $\mathsf{AT}(\Pi)$. The set of all worlds of Π will be denoted by $\mathcal{W}_\Pi$.

A *model* of a clause $\Gamma \rightarrow \mathsf{p}$ is a world $W \in \mathcal{W}_\Pi$ such that either $\Gamma \not\subseteq W$ or $\Gamma \cup \{\mathsf{p}\} \subseteq W$. A model of a set of clauses Δ is a model of all its clauses. Such a model not necessarily exists. For instance, take the KB consisting on the facts $\rightarrow \mathsf{p}$ and $\rightarrow \overline{\mathsf{p}}$.

The semantics of a KB Δ will be given by its *invariant*. The atoms that belong to all models of Δ constitute its invariant, denoted by $\mathcal{J}(\Delta)$. Atoms belonging to the invariant of Δ are said to be *bound* in Δ. A proposition $p \in \Pi$ is said to be *free* in Δ if there are models W_1, W_2 of Δ such that $\mathsf{p} \in W_1$ and $\overline{\mathsf{p}} \in W_2$.

We are interested in the constructoin of the invariant of a set of clauses. The sequent system proposed in the next section achieves this goal.

3 A Sequent System for Belief Databases

The system will prove properties of tuples $\langle \Delta, S \rangle$, where Δ is a KB and S is a set of atoms. Such a tuple will be said to be *consistent* iff there is some model W of Δ such that $S \subseteq W$. A *model* of a tuple $\langle \Delta, S \rangle$ is a model of Δ which contains S.

The system B_0 is given by the following rules.

$$\mathsf{id}\ \frac{}{\langle \Delta, \emptyset \rangle} \qquad\qquad \mathsf{inf}\ \frac{\langle \Delta \cup \{\mathsf{p}\}, S \rangle}{\langle \Delta, C, S \cup \{\mathsf{p}\} \rangle}$$

$$\mathsf{te1}\ \frac{\langle \Delta \cup \{\Gamma, \mathsf{p} \to \mathsf{q}\}, S\rangle \qquad \mathsf{p} \in S}{\langle \Delta \cup \{\Gamma \to \mathsf{q}\}, S\rangle} \qquad \mathsf{te2}\ \frac{\langle \Delta \cup \{\Gamma \to \mathsf{p}\}, S\rangle \qquad \mathsf{p} \in S}{\langle \Delta, S\rangle}$$

$$\mathsf{ce1}\ \frac{\langle \Delta \cup \{\Gamma, \overline{\mathsf{q}} \to \overline{\mathsf{p}}\}, S\rangle \qquad \mathsf{p} \in S}{\langle \Delta \cup \{\Gamma \to \mathsf{q}\}, S\rangle} \qquad \mathsf{ce2}\ \frac{\langle \Delta \cup \{\Gamma, \overline{\mathsf{p}} \to \mathsf{q}\}, S\rangle \qquad \mathsf{p} \in S}{\langle \Delta, S\rangle}$$

$$\perp\ \frac{\langle \Delta \cup \{\overline{\mathsf{p}}\}, S\rangle \qquad \mathsf{p} \in S}{\langle \emptyset, \mathsf{AT}(\Pi)\rangle} \qquad \mathsf{rec}\ \frac{\langle \Delta, \rangle \qquad \begin{array}{c}\langle \Delta \cup \{\mathsf{p}\}, S\rangle \\ \vdots \\ \langle \Delta_1, S_1\rangle\end{array} \qquad \begin{array}{c}\langle \Delta \cup \{\overline{\mathsf{p}}\}, S\rangle \\ \vdots \\ \langle \Delta_2, S_2\rangle\end{array}}{\langle \Delta, S_1 \cap S_2\rangle}$$

Figure 1: The system B_0.

We add the proviso that rule rec may be applied only once for each pair of complementary atoms. We assume further that in rule rec no rules may be applied to the tuples $\langle \Delta_1, S_1\rangle$ and $\langle \Delta_2, S_2\rangle$. A rule that may be applied is said to be *allowed*. The proviso is to avoid a case like the following one:

$\mathsf{p} \quad \to \quad \mathsf{q}$
$\overline{\mathsf{p}} \quad \to \quad \mathsf{r}$

Here rule rec might be applied eternally had we not imposed the above condition.

4 Soundness

If Δ is a KB, a *proof* in system B from Δ is a finite sequence of tuples

$$\langle \Delta_0, C_0, S_0\rangle, \ldots, \langle \Delta_n, C_n, S_n\rangle$$

such that: (1) $\Delta_0 = \Delta$, (2) $S_0 = \emptyset$, and (3) for all $0 \le j < n$, tuple $j+1$ is obtained by application of some allowed rule of B on tuple j.

A proof is linear except in the case of the rule rec, the only case where the proof branches out, producing two sub-proofs. In both cases, all allowed rules must be applied until no one is allowed. Thus, any other occurrence of rule rec must occur within the sub-proof of another application of the same rule. The following figure shows this.

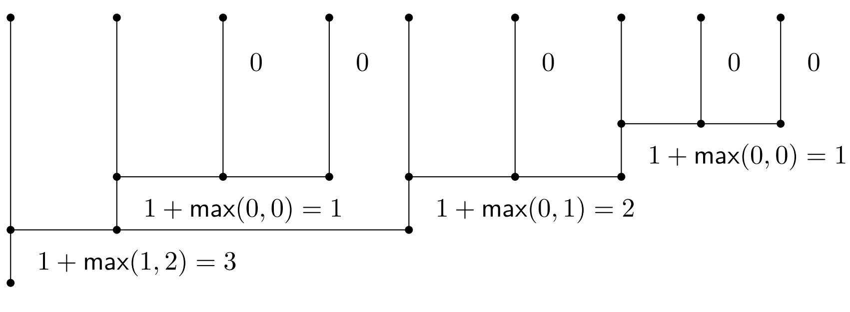

Figure 2: The general structure of a proof.

We define inductively the *depth* of a proof as follows:

- The depth of a proof without any application of rule rec is 0.

- The depth of a proof with an application of rule rec is one plus the maximum of the depths of each sub-proof. The depth of the proof schematised in figure 2 is thus 3.

The depth of a proof will provide the induction schema for the proofs.

Lemma 1 *Let Δ be a KB without any occurrence of complementary atoms. Then Δ is consistent.*
Proof. *Since no complementary atoms occur in Δ, any maximal consistent set $W \subseteq \mathcal{W}_\Pi$ containing all atoms that occur in Δ is a model of Δ, since all clauses will be set to the tautology $\mathsf{true} \rightarrow \mathsf{true}$.*

□

Corollary 1 *Let $\langle \Delta, S \rangle$ be a tuple such that Δ has no occurrence of complementary atoms, S is consistent and none of the atoms of S occurs in Δ. Then the tuple is consistent.*
Proof. *Since no atoms of S occur in Δ, it is always possible to construct a maximal consistent set containing S and all atoms in Δ.*

□

Lemma 2 *Let Δ be a consistent KB based on Π and let $W \subseteq \mathcal{W}_\Pi$. Then W is a model of Δ iff for all $\mathsf{p} \in W$, it is a model of $\Delta \cup \{\rightarrow \mathsf{p}\}$.*
Proof. *It is clear that a model of $\Delta \cup \{\rightarrow \mathsf{p}\}$ is a model of Δ. If W is a model of Δ that contains p, it is also a model of the clause $\rightarrow \mathsf{p}$. Thus, it is a model of $\Delta \cup \{\rightarrow \mathsf{p}\}$.*

□

Lemma 3 *(1) A world W is a model of $\langle \Delta \cup \{\rightarrow \mathsf{p}\}, S\rangle$ iff it is a model of $\langle \Delta, S \cup \{\mathsf{p}\}\rangle$. (2) A world W is a model of $\langle \Delta \cup \{\Gamma, \mathsf{p} \rightarrow \mathsf{q}\}, S \cup \{\mathsf{p}\}\rangle$ iff it is a model of $\langle \Delta \cup \{\Gamma \rightarrow \mathsf{q}\}, S \cup \{\mathsf{p}\}\rangle$. (3) A world W is a model of $\langle \Delta \cup \{\Gamma \rightarrow \mathsf{p}\}, S \cup \{\mathsf{p}\}\rangle$ iff it is a model of $\langle \Delta, S \cup \{\mathsf{p}\}\rangle$. (4) A world W is a model of $\langle \Delta \cup \{\Gamma, \overline{\mathsf{q}} \rightarrow \overline{\mathsf{p}}\}, S \cup \{\mathsf{p}\}\rangle$ iff it is a model of $\langle \Delta \cup \{\Gamma \rightarrow \mathsf{q}\}, S \cup \{\mathsf{p}\}\rangle$. (5) A world W is a model of $\langle \Delta \cup \{\Gamma, \overline{\mathsf{p}} \rightarrow \mathsf{q}\}, S \cup \{\mathsf{p}\}\rangle$ iff it is a model of $\langle \Delta, S \cup \{\mathsf{p}\}\rangle$.*
Proof. *For 1, we have by lemma 2 that any model of $\Delta \cup \{\rightarrow \mathsf{p}\}$ contains p and any model of Δ containing $S \cup \{\mathsf{p}\}$ is a model of $\Delta \cup \{\rightarrow \mathsf{p}\}$.*

For 2, we have that any model of $\Delta \cup \{\Gamma, \mathsf{p} \rightarrow \mathsf{q}\}$ containing $S \cup \{\mathsf{p}\}$ is a model of $\Delta \cup \{\Gamma \rightarrow \mathsf{q}\}$ and any model of $\Delta \cup \{\Gamma \rightarrow \mathsf{q}\}$ containing $S \cup \{\mathsf{p}\}$ is a model of $\Delta \cup \{\Gamma \rightarrow \mathsf{q}\}$.

For 3, we have that any model of $\Delta \cup \{\Gamma \rightarrow \mathsf{p}\}$ containing $S \cup \{\mathsf{p}\}$ is a model of Δ and any model of Δ containing $S \cup \{\mathsf{p}\}$ is a model of the clause $\Gamma \rightarrow \mathsf{p}$.

For 4, any model of $\Delta \cup \{\Gamma, \overline{\mathsf{q}} \rightarrow \overline{\mathsf{p}}\}$ containing $S \cup \{\mathsf{p}\}$ must imply that either Γ or $\overline{\mathsf{q}}$ are false. Thus, it must be a model of the clause $\Gamma \rightarrow \mathsf{q}$. Conversely, any model of $\Delta \cup \{\Gamma \rightarrow \mathsf{q}\}$ containing $S \cup \{\mathsf{p}\}$ is a model of the clause $\Gamma, \overline{\mathsf{q}} \rightarrow \overline{\mathsf{p}}$.

For 5, any model of $\Delta \cup \{\Gamma, \overline{\mathsf{p}} \rightarrow \mathsf{q}\}$ is a model of Δ; conversely, any model of Δ containing $S \cup \{\mathsf{p}\}$ is a model of the clause $\Gamma, \overline{\mathsf{p}} \rightarrow \mathsf{q}$.

□

Lemma 3 states that rules inf, $\mathsf{te1}$, $\mathsf{te2}$, $\mathsf{ce1}$, and $\mathsf{ce2}$ preserve consistency (and inconsistency.) Now we will analyse rule rec. We need some previous results first.

Lemma 4 *Let Δ be an inconsistent KB, and let $\langle \Delta_0, S_0 \rangle, \ldots, \langle \Delta_n, S_n \rangle$ be a proof in B such that $\Delta_0 = \Delta$, $S_0 = \emptyset$ and no further rules are allowed in the last tuple. Then $S_n = \mathsf{AT}(\Pi)$.*

Proof. *Induction on the depth of the proof. By lemma 1, an inconsistent KB must have at least one pair of complementary atoms.*

Base case: *The depth of the proof is 1. So there is one application of rule* rec *on* $\langle \Delta_k, S_k \rangle$*, giving rise to two sub-proofs, one beginning with* $\langle \Delta_k \cup \{\mathsf{p}\}, S_k \rangle$ *and the other one with* $\langle \Delta_k \cup \{\overline{\mathsf{p}}\}, S_k \rangle$*. Since by assumption* Δ_0 *is inconsistent and we so far have applied only rules* inf, $\mathsf{ce1}$, $\mathsf{ce2}$, $\mathsf{te1}$, *and* $\mathsf{te2}$, $\langle \Delta_k, S_k \rangle$ *must be also inconsistent. Thus, both sub-proofs begin with inconstent tuples. Let us consider one of the sub-proofs (the same considerations apply to the other one.) By assumption the depth of the sub-proof is 0. Thus, only rules* inf, $\mathsf{ce1}$, $\mathsf{ce2}$, $\mathsf{te1}$, $\mathsf{te2}$, *and* $\perp$ *may be applied. Since* p *and* $\overline{\mathsf{p}}$ *belong both to* $\Delta_k \cup \{\mathsf{p}\}$*, then these rules try to eliminate all occurrences of the complementary atoms. If* $\overline{\mathsf{p}}$ *is eliminated using rule* $\mathsf{ce2}$*, we would get a set of clauses without any occurrence of complementary atoms and a set of atoms not occurring in the clauses. This is consistent, contradicting thus the assumption. So, the only possibility is to apply rule* $\mathsf{ce2}$.

Assume we rewrite a clause $\Gamma, \overline{\mathsf{q}} \rightarrow \overline{\mathsf{p}}$ *as* $\Gamma \rightarrow \mathsf{q}$*. If as a result we get a new pair of complementary atoms* q, $\overline{\mathsf{q}}$*, we contradict the assumption that the depth of the sub-proof is 0, since rule* rec *would be applicable on the new pair of complementary atoms. Otherwise, we would have eliminated* $\overline{\mathsf{p}}$ *and we would have again a set of clauses without any occurrence of complementary atoms and a set of atoms not occurring in the set. So, the only possibility left is that we have a clause* $\rightarrow \overline{\mathsf{p}}$ *and rule* $\perp$ *is thus allowed.*

Induction step: *assume the statement holds for a proof of depth* m *and we have a proof of depth* $m+1$*. Assume we apply rule* rec *on* $\langle \Delta_k, S_k \rangle$*, giving thus rise to two sub-proofs beginning with* $\langle \Delta_k \cup \{\mathsf{p}\}, S_k \rangle$ *and* $\langle \Delta_k \cup \{\overline{\mathsf{p}}\}, S_k \rangle$ *and ending with* $\langle \Delta_1, S_1 \rangle$ *and* $\langle \Delta_2, S_2 \rangle$ *respectively. Since we have so far applied only rules* inf, $\mathsf{ce1}$, $\mathsf{ce2}$, $\mathsf{te1}$, *and* $\mathsf{te2}$*, the tuple* $\langle \Delta k, S_k \rangle$ *is inconsistent and so are* $\langle \Delta_k \cup \{\mathsf{p}\}, S_k \rangle$ *and* $\langle \Delta_k \cup \{\overline{\mathsf{p}}\}, S_k \rangle$*. By assumption the depth of both sub-proofs is* m *(or less) and thus* $S_1 = S_2 = S_1 \cap S_2 = \mathsf{AT}(\Pi)$.

$\square$

Lemma 5 *Let* Δ*be a consistent set of clauses and let* $\langle \Delta_0, S_0 \rangle \ldots, \langle \Delta_n, S_n \rangle$ *be a proof such that (1)* $\Delta_0 = \Delta$*, and (2)* $\S_0 = \emptyset$*. Then for any tuple* $\langle \Delta_k, S_k \rangle$ *occurring in the proof, a set of atoms* M *is a model of* $\langle \Delta_n, S_n \rangle$ *iff it is a model of any tuple* $\langle \Delta_k, S_k \rangle$*,* $0 \leq k \leq n$*, occurring in the proof. In particular, iff it is a model of* Δ.

Proof. *Induction on the depth of the proof.*

Base case: *Assume the depth of the proof is 0. Then, only rules* inf, $\mathsf{ce1}$, $\mathsf{ce2}$, $\mathsf{te1}$, *and* $\mathsf{te2}$ *may be applied. The result follows from lemma 3.*

Induction step: *Assume the statement holds for proofs with depth* m*. If our proof has depth* $m+1$*, then after some applications of rules* inf, $\mathsf{ce1}$, $\mathsf{ce2}$, $\mathsf{te1}$, *and* $\mathsf{te2}$*, rule* rec *will be applied on a tuple* $\langle \Delta_k, S_k \rangle$ *giving rise to two sub-proofs beginning with* $\langle \Delta_k \cup \{\rightarrow \mathsf{p}\}, S_k \rangle$ *and* $\langle \Delta_k \cup \{\rightarrow \overline{\mathsf{p}}\}, S_k \rangle$ *and ending with* $\langle \Delta_1, S_1 \rangle$ *and* $\langle \Delta_2, S_2 \rangle$ *respectively. Since we had so far applied only rules other than* rec, $\langle \Delta_k, S_k \rangle$ *is consistent by lemma 3. We have here two possibilities: either* p *is bound in* Δ_k *or it is free therein.*

Is p *is bound in* Δ_k*, then* $\Delta_k \cup \{\rightarrow \overline{\mathsf{p}}\}$ *is inconsistent and by lemma 4,* $S_2 = \mathsf{AT}(\Pi)$*. Thus* $S_1 \cap S_2 = S_1$*. Besides, by assumption the depth of the sub-proof is* m *or less, and by induction hypothesis, a set of atoms* M *is a model of* Δ_1 *containing* S_1 *iff it is a model of* $\Delta_k \cup \{\rightarrow \mathsf{p}\}$ *containing* S_k*. Since* p *is bound in* Δ_k*, all models of* Δ_k *contain* p *and thus,* M *is a model of* $\langle \Delta_1, S_1 \rangle$ *iff it is a model of* Δ.

If p *is free in* Δ_k*, then both* $\Delta_k \cup \{\rightarrow \mathsf{p}\}$ *and* $\Delta_k \cup \{\rightarrow \overline{\mathsf{p}}\}$ *are consistent. Since by assumption the depth of the sub-proofs is* m *or less, by induction hypothesis we have that a set* M_1 *is a model of* Δ_1 *that contains* S_1 *iff it is a model of* $\Delta_k \cup \{\rightarrow \mathsf{p}\}$ *that contains* S_k*. In the same way, a set* M_2 *is a model of* Δ_2 *that contains* S_2 *iff* *it is a model of $\Delta_k \cup \{\to \overline{\mathsf{p}}\}$ that contains S_k. Thus, a set M is a model of Δ_k that contains S_k iff it is a model of Δ_k that contains $S_1 \cap S_2$.*

$\square$

Theorem 1 (Soundness) *: Let Δ be a consistent KB and let $\langle \Delta', S\rangle$ be any tuple occurring in a proof beginning with $\langle \Delta, \emptyset$. Then all atoms in S are bound in Δ.*
Proof. *By lemmas 3 and 5, any model of Δ is a model of Δ' containing S. Thus all models of Δ contain S.*

$\square$

5 Completeness

In section 4 we showed that for any KB Δ, a proof will end in $\langle \Delta', \mathsf{AT}(\Pi)\rangle$ if Δ is inconsistent and in $\langle \Delta', S\rangle$ if it is consistent, where all atoms in S are bound in Δ. Now we will show that if no further rules are applicable on $\langle \Delta', S\rangle$, then S contains all atoms that are bound in Δ.

Theorem 2 (Completeness) *Let Δ be a consistent KB and let $\langle \Delta', S\rangle$ be the last tuple of a a proof such that no further rules are applicable on it. Then all atoms that are bound in Δ are in S.*
Proof. *Induction on the depth of the proof.*
Base case: *The depth of the proof is 0. Thus, the proof did not apply the* rec *rule and there are no complementary atoms in Δ'. Since for all atoms $\mathsf{p} \in S$, neither* p *nor $\overline{\mathsf{p}}$ occur in Δ' (otherwise at least one of the rules* te1, te2, ce1, *or* ce2 *would be allowed), we can set all atoms occurring in Δ' to* true *or to* false. *In both cases, we "complete" a maximal consistent set with the atoms in S and we get two models of Δ'. Thus all propositions occurring in Δ' are free.*
Induction step: *Assume that the statement holds for proofs which have a depth m. If the proof has a depth of $m + 1$, we will have some applications of rules* inf, ce1, ce2, te1, *and* te2 *until rule* rec *is applied on $\langle \Delta_k, S_k\rangle$, giving rise to two subproofs beginning with $\langle \Delta_k \cup \{\to \mathsf{p}\}, S_k\rangle$ and $\langle \Delta_k \cup \{\to \overline{\mathsf{p}}\}, S_k\rangle$ and ending with $\langle \Delta_1, S_1\rangle$ and $\langle \Delta_2, S_2\rangle$ respectively. Both sub-proofs will have depth m. So the induction hypothesis may be applied. Hence, all atoms that are bound in $\Delta_k \cup \{\to \mathsf{p}\}$ are in S_1 and all atoms that are bound in $\Delta_k \cup \{\to \overline{\mathsf{p}}\}$ are in S_2. Thus, all models of $\Delta_k \cup \{\to \mathsf{p}\}$ contain S_1 and all models of $\Delta_k \cup \{\to \overline{\mathsf{p}}\}$ contain S_2. Therefore all models of Δ contain $S_1 \cap S_2$.*

$\square$

6 Conclusions

The sequent system shown here allows the construction of the invariant of a set of propositional clauses. The system is consistent and complete and can be implemented recursively (for instance, as a Prolog[1] program.) A proof always terminates, since the application of rule rec can only occur finitely many times and other rules are either terminal (in the case of $\perp$), or they eliminate one atom (ce1 or te1) or one clause (inf, ce2, or te2.) The complexity may grow exponentially with the number of complementary atoms.

References

[1] Lloyd, J.W.: *Foundations of Logic Programming*, Springer Verlag, 1987.