Partial results about partial cut elimination for logics of common knowledge, part I

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Abstract

The notions of *common knowledge* or *common belief* play an important role in several areas of computer science (e.g. distributed systems, communication), in philosophy, game theory, artificial intelligence, psychology and many other fields which deal with the interaction within a group of "agents", agreement or coordinated actions. In the following we will present several deductive systems for common knowledge above epistemic logics – such as **K**, **T**, **S4** and **S5** – with a fixed number of agents. We focus on structural and proof-theoretic properties of these calculi.

1 Introduction

The notions of *common knowledge* or *common belief* play an important role in several areas of computer science (e.g. distribute systems, communication), in philosophy, game theory, artificial intelligence, psychology and many other fields which deal with the interaction within a group of "agents", agreement or coordinated actions. Everybody has a vague intuitive understanding of what common knowledge (belief) should be, and for a lot of applications such informal approaches may suffice. On the other hand, in many cases a formal mathematical treatment of common knowledge (belief) is required.

There are two main directions in developing formalizations of reasoning with and about common knowledge:

- Barwise (cf. e.g. [1, 2]) discusses common knowledge within his *Situation Semantics* and his general treatment *Situation in Logic*. Basic ingredients are the sets SIT of situations and FACTS of facts.

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- Alternatively, common knowledge may also be studied starting off from epistemic logics, i.e. in the context of multi-modal logics; see the textbooks Fagin, Halpern, Moses and Vardi [3] and Meyer and van der Hoek [8] for a good introduction.

Although being built up from different "atoms", there exist interesting connections between these two formal frameworks for common knowledge. For example, largest fixed points of suitable operators are used in a crucial way in both cases. In this article, however, we will confine ourselves to common knowledge in its multi-modal version. More about its relationship to common knowledge à la Barwise can be found in Graf [5] and Lismont [7].

In the following we will present several deductive systems for common knowledge above epistemic logics – such as **K**, **T**, **S4** and **S5** – with a fixed number of agents. We focus on structural and proof-theoretic properties of these calculi, also with respect to efficient implementations of logics of common knowledge, which will be studied elsewhere.

For completeness we begin with presenting the syntax and semantics of our logics of common knowledge and introduce their standard Hilbert-style formulations. In the later sections we turn to the corresponding Tait-calculi and study several forms of partial cut elimination.

2 Syntax and semantics of logics of common knowledge

Let $L_n(\mathsf{C})$ be our standard language for multi-modal logic which comprises a set PROP of *atomic propositions*, typically indicated by $P, Q, \dots$ (possibly with subscripts), the *propositional connectives* $\lor$ and $\land$, the *epistemic operators* $\mathsf{K}_1, \mathsf{K}_2, \dots, \mathsf{K}_n$ and the *common knowledge operator* C; in addition we assume that there is an auxiliary symbol $\sim$ for forming the complements of atomic propositions and complementary epistemic operators. The *formulas* $\alpha, \beta, \gamma, \dots$ (possibly with subscripts) of $L_n(\mathsf{C})$ and the *complexity measure* $\mathsf{me}(\alpha)$ for each $L_n(\mathsf{C})$ formula α are inductively generated as follows:

1. P and $\sim P$ are $L_n(\mathsf{C})$ formulas; $\mathsf{me}(P) := \mathsf{me}(\sim P) := 0$.

2. If α and β are $L_n(\mathsf{C})$ formulas, so are $(\alpha \lor \beta)$ and $(\alpha \land \beta)$;
$$\mathsf{me}(\alpha \lor \beta) := \mathsf{me}(\alpha \land \beta) := \max(\mathsf{me}(\alpha), \mathsf{me}(\beta)) + 1.$$

3. If α is an $L_n(\mathsf{C})$ formula, so are $\mathsf{K}_i(\alpha)$ and $\sim\mathsf{K}_i(\alpha)$;
$$\mathsf{me}(\mathsf{K}_i(\alpha)) := \mathsf{me}(\sim\mathsf{K}_i(\alpha)) := \mathsf{me}(\alpha) + 1.$$

4. If α is an $L_n(\mathsf{C})$ formula, so are $\mathsf{C}(\alpha)$ and $\sim\mathsf{C}(\alpha)$;

$$\mathsf{me}(\mathsf{C}(\alpha)) \;:=\; \mathsf{me}(\sim\mathsf{C}(\alpha)) \;:=\; \mathsf{me}(\alpha) + n + 1.$$

See below for an explanation why the number n of agents has to be added in the last clause. The $L_n(\mathsf{C})$ formulas $\sim P$, $\sim\mathsf{K}_i(\alpha)$ and $\sim\mathsf{C}(\alpha)$ act as negations of the atomic proposition P and the modal atoms $\mathsf{K}_i(\alpha)$ and $\mathsf{C}(\alpha)$, respectively. The negations $\neg\alpha$ of general $L_n(\mathsf{C})$ formulas α are defined by making use of de Morgan's laws and the law of double negation:

1. If α is the atomic proposition P, then $\neg\alpha$ is $\sim P$; if α is the formula $\sim P$, then $\neg\alpha$ is P.
2. If α is the formula $(\beta \vee \gamma)$, then $\neg\alpha$ is $(\neg\beta \wedge \neg\gamma)$; if α is the formula $(\beta \wedge \gamma)$, then $\neg\alpha$ is $(\neg\beta \vee \neg\gamma)$.
3. If α is the formula $\mathsf{K}_i(\beta)$, then $\neg\alpha$ is $\sim\mathsf{K}_i(\neg\beta)$; if α is the formula $\sim\mathsf{K}_i(\beta)$, then $\neg\alpha$ is $\mathsf{K}_i(\neg\beta)$.
4. If α is the formula $\mathsf{C}(\beta)$, then $\neg\alpha$ is $\sim\mathsf{C}(\neg\beta)$; if α is the formula $\sim\mathsf{C}(\beta)$, then $\neg\alpha$ is $\mathsf{C}(\neg\beta)$.

Often we omit parentheses if there is no danger of confusion and abbreviate the remaining logical connectives as usual; in addition we set

$$\mathsf{E}(\alpha) \;:=\; \mathsf{K}_1(\alpha) \wedge \ldots \wedge \mathsf{K}_n(\alpha).$$

The definition of the complexity measure of the formulas $\mathsf{C}(\alpha)$ and $\sim\mathsf{C}(\alpha)$ has been tailored so that we always have

$$\mathsf{me}(\mathsf{E}(\alpha)) \;=\; \mathsf{me}(\sim\mathsf{E}(\alpha)) \;<\; \mathsf{me}(\mathsf{C}(\alpha)) \;=\; \mathsf{me}(\sim\mathsf{C}(\alpha)).$$

A possible intuitive interpretation of $\mathsf{K}_i(\alpha)$ is "agent i knows (believes) that α", and thus $\mathsf{E}(\alpha)$ can be understood as "everybody knows (believes) that α". The latter formula has to be strictly distinguished from $\mathsf{C}(\alpha)$, which expresses common knowledge of α among the agents 1 to n (see below). We also need the iterations $\mathsf{E}^m(\alpha)$ for all natural numbers m, inductively introduced as

$$\mathsf{E}^0(\alpha) \;:=\; \alpha \quad \text{and} \quad \mathsf{E}^{m+1}(\alpha) \;:=\; \mathsf{E}(\mathsf{E}^m(\alpha)).$$

Turning to the semantics of $L_n(\mathsf{C})$, we define a *Kripke-frame* (for $L_n(\mathsf{C})$) to be an $(n{+}1)$-tuple

$$\mathfrak{M} \;=\; (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$$

for a non-empty set W of *worlds* and binary relations $\mathbb{K}_1, \dots, \mathbb{K}_n$ on W; the set of worlds of a Kripke-frame $\mathfrak{M}$ is often denoted by $|\mathfrak{M}|$. A *valuation* in $\mathfrak{M}$ then is a function $\mathcal{V}$ from the atomic propositions PROP to the power set $Pow(|\mathfrak{M}|)$ of $|\mathfrak{M}|$,

$$\mathcal{V} \;:\; \mathsf{PROP} \;\to\; Pow(|\mathfrak{M}|).$$

The *truth-set* $\|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}$ of an $L_n(\mathsf{C})$ formula α with respect to the Kripke-frame $\mathfrak{M} = (W, \mathbb{K}_1, \dots, \mathbb{K}_n)$ and a valuation $\mathcal{V}$ is defined, as usual in multi-modal logics, by induction an the complexity of α with an additional clause for treating the operator C:

$$\begin{array}{rcl}
\|P\|^{\mathfrak{M}}_{\mathcal{V}} & := & \mathcal{V}(P), \\
\|{\sim}P\|^{\mathfrak{M}}_{\mathcal{V}} & := & W \setminus \|P\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\alpha \lor \beta\|^{\mathfrak{M}}_{\mathcal{V}} & := & \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \cup \|\beta\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\alpha \land \beta\|^{\mathfrak{M}}_{\mathcal{V}} & := & \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \cap \|\beta\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\mathsf{K}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} & := & \{\, v \in W : w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \text{ for all } (v, w) \in \mathbb{K}_i \,\}, \\
\|{\sim}\mathsf{K}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} & := & W \setminus \|\mathsf{K}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} & := & \bigcap \{\, \|\mathsf{E}^m(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} : m \geq 1 \,\}, \\
\|{\sim}\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} & := & W \setminus \|\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}}.
\end{array}$$

By means of these truth-sets we can easily express that the $L_n(\mathsf{C})$ formula α is valid in the Kripke-frame $\mathfrak{M}$ with respect to valuation $\mathcal{V}$ and world w; this is the case if $w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}$. The following notation is convenient for expressing this situation:

$$(\mathfrak{M}, \mathcal{V}, w) \models \alpha \quad :\Longleftrightarrow \quad w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}.$$

Observe that these semantics do not imply that α is true in all worlds which satisfy $\mathsf{C}(\alpha)$. In the literature sometimes a distinction is made between knowledge and belief: knowledge of a fact implies the truth of this fact, whereas the belief of something may be compatible with its falsity. But since the intuitive meaning of knowledge or belief can only be approximated and can never be completely grasped by formal semantics, we will not pay attention to this subtlety.

If we have $(\mathfrak{M}, \mathcal{V}, w) \models \alpha$ for all valuations $\mathcal{V}$ in $\mathfrak{M}$ and all worlds $w \in |\mathfrak{M}|$ of a Kripke-frame $\mathfrak{M}$, then α is *valid* in $\mathfrak{M}$,

$$\mathfrak{M} \models \alpha.$$

Our semantics reflects the so-called *iterative interpretation* of common knowledge:

$$(\mathfrak{M}, \mathcal{V}, w) \models \mathsf{C}(\alpha) \quad \iff \quad (\mathfrak{M}, \mathcal{V}, w) \models \bigwedge_{m \geq 1} \mathsf{E}^m(\alpha).$$

Thus α is common knowledge if everybody knows α and everybody knows that everybody knows α and everybody knows that everybody knows that everybody knows α and so on. Alternatively, we could also treat common knowledge in the sense of the *greatest fixed point interpretation* since

$$(\star) \qquad \|\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} \;=\; \bigcup \{\, X \subset |\mathfrak{M}| : X = \|\mathsf{E}(\alpha) \wedge \mathsf{E}(Q)\|^{\mathfrak{M}}_{\mathcal{V}[Q:=X]} \,\}$$

where Q is chosen to be an atomic proposition which does not occur in α and $\mathcal{V}[Q := X]$ is the valuation which maps Q to X and otherwise agrees with $\mathcal{V}$. A proof of equation $(\star)$ can be found, for example, in Fagin, Halpern, Moses and Vardi [3].

Property $(\star)$ follows from the continuity of the operator defined by the formula $(\mathsf{E}(\alpha) \wedge \mathsf{E}(Q))$. There are variants of common knowledge like ϵ-common knowledge or $\diamond$-common knowledge so that $\mathsf{C}^\epsilon(\alpha)$ and $\mathsf{C}^\diamond(\alpha)$ cannot be characterized by the union of the *finite* iterations of the corresponding operators; then only the greatest fixed point approach makes sense (cf. e.g. [6, 3]).

Now we recall the Hilbert-style formulations of a few multi-modal logics of common knowledge. We begin with the usual logic **K**, extended to n agents plus C, and denote it by $\mathbf{K}_n(\mathsf{C})$.

Basic axioms of $\mathbf{K}_n(\mathsf{C})$

(TAUT) All propositional tautologies

(K) $\mathsf{K}_i(\alpha) \wedge \mathsf{K}_i(\alpha \to \beta) \;\to\; \mathsf{K}_i(\beta)$

Basic rules of inference of $\mathbf{K}_n(\mathsf{C})$

(MP) $$\frac{\alpha \qquad \alpha \to \beta}{\beta}$$

(NEC) $$\frac{\alpha}{\mathsf{K}_i(\alpha)}$$

Co-closure axioms of $\mathbf{K}_n(\mathsf{C})$

(CCL) $\mathsf{C}(\alpha) \;\to\; (\mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha)))$

Induction rules of $\mathbf{K}_n(\mathsf{C})$

(IND) $$\frac{\beta \ \rightarrow \ \mathsf{E}(\alpha) \wedge E(\beta)}{\beta \ \rightarrow \ \mathsf{C}(\alpha)}$$

In these axioms and rules and in the ones which will be formulated below, α and β may be arbitrary $L_n(\mathsf{C})$ formulas. The system $\mathbf{T}_n(\mathsf{C})$ is obtained from $\mathbf{K}_n(\mathsf{C})$ by adding all axioms

(T) $$\mathsf{K}_i(\alpha) \ \rightarrow \ \alpha.$$

$\mathbf{S4}_n(\mathsf{C})$ is the multi-modal version of **S4** with common knowledge and extends $\mathbf{T}_n(\mathsf{C})$ by all axioms (4) for positive introspection

(4) $$\mathsf{K}_i(\alpha) \ \rightarrow \ \mathsf{K}_i(\mathsf{K}_i(\alpha)).$$

Finally, adding the corresponding axioms (5) of negative introspection to the theory $\mathbf{S4}_n(\mathsf{C})$ gives the system $\mathbf{S5}_n(\mathsf{C})$,

(5) $$\neg\mathsf{K}_i(\alpha) \ \rightarrow \ \mathsf{K}_i(\neg\mathsf{K}_i(\alpha)).$$

Now let $\mathbf{F}$ be one of the theories $\mathbf{K}_n(\mathsf{C})$, $\mathbf{T}_n(\mathsf{C})$, $\mathbf{S4}_n(\mathsf{C})$ or $\mathbf{S5}_n(\mathsf{C})$. We employ the standard notion of provability of an $L_n(\mathsf{C})$ formula α in the theory $\mathbf{F}$ and write this fact as

$$\mathbf{F} \vdash \alpha.$$

A Kripke-frame $\mathfrak{M}$ is a *model* of $\mathbf{K}_n(\mathsf{C})$ if all axioms of $\mathbf{K}_n(\mathsf{C})$ are valid in $\mathfrak{M}$ and if $\mathfrak{M}$ is closed under the rules of inference of $\mathbf{K}_n(\mathsf{C})$ with respect to validity. And a standard result of modal logic characterizes the Kripke-frames $\mathfrak{M} = (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$ which are models of these theories:

(1) $\mathfrak{M}$ is a model of $\mathbf{K}_n(\mathsf{C})$ for arbitrary (binary) $\mathbb{K}_1, \ldots, \mathbb{K}_n$.

(2) $\mathfrak{M}$ is a model of $\mathbf{T}_n(\mathsf{C})$ if and only if the $\mathbb{K}_1, \ldots, \mathbb{K}_n$ are reflexive.

(3) $\mathfrak{M}$ is a model of $\mathbf{S4}_n(\mathsf{C})$ if and only if the $\mathbb{K}_1, \ldots, \mathbb{K}_n$ are reflexive and transitive.

(4) $\mathfrak{M}$ is a model of $\mathbf{S5}_n(\mathsf{C})$ if and only if the $\mathbb{K}_1, \ldots, \mathbb{K}_n$ are equivalence relations.

Following the standard patterns, we call the $L_n(\mathsf{C})$ formula α a *semantic consequence* of $\mathbf{K}_n(\mathsf{C})$,

$$\mathbf{K}_n(\mathsf{C}) \models \alpha,$$

if α is valid in all models of $\mathbf{K}_n(\mathsf{C})$. The subsequent theorem states that syntactic derivability is adequate for semantic consequence in all our logics.

Theorem 1 (Soundness and completeness) *Let* $\mathbf{F}$ *be one of the logics* $\mathbf{K}_n(\mathsf{C})$*,* $\mathbf{T}_n(\mathsf{C})$*,* $\mathbf{S4}_n(\mathsf{C})$ *or* $\mathbf{S5}_n(\mathsf{C})$*. Then we have*

$$\mathbf{F} \vdash \alpha \quad \Longleftrightarrow \quad \mathbf{F} \models \alpha.$$

Let us now come back to the co-closure axioms and induction rules of, say, $\mathbf{K}_n(\mathsf{C})$. The axiom (CCL) states that each formula $\mathsf{C}(\alpha)$ describes a set of states co-closed under the operator

$$\mathrm{Op}_\alpha(X) \;:=\; \mathsf{E}(\alpha) \wedge \mathsf{E}(X)$$

mapping sets of states to sets of states, with respect to a given frame. The rules (IND), on the other hand, formulate, that $\mathsf{C}(\alpha)$ is the greatest (definable) set co-closed under Op_α. So we immediately obtain that $\mathsf{C}(\alpha)$ is the largest fixed point of Op_α, i.e.

$$\mathbf{K}_n(\mathsf{C}) \;\vdash\; \mathsf{C}(\alpha) \;\leftrightarrow\; \mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha)).$$

Proof-theoretic experience should provide a clear indication that the interplay of (CCL) and (IND) may cause serious difficulties in finding good deductive systems for $\mathbf{K}_n(\mathsf{C})$ and the other multi-modal logics mentioned before.

3 A Tait-style reformulation of $\mathbf{K}_n(\mathsf{C})$

In this and the following sections we will look more carefully at the deductive and procedural aspects of our logics of common knowledge. For simplicity we restrict ourselves to the theory $\mathbf{K}_n(\mathsf{C})$ and leave the other logics to the reader.

Obviously, inference rules like modus ponens (MP), which violate the subformula property, make reasonable backward proof search impossible. The first steps thus are a reformulation of $\mathbf{K}_n(\mathsf{C})$ as a Tait-style system with cuts and an attempt to "tame" general cuts in a suitable way.

In the Tait-calculus $\overline{\mathbf{K}}_n(\mathsf{C})$ derives finite sets of $L_n(\mathsf{C})$ formulas which have to be interpreted disjunctively. The capital Greek letters $\Gamma, \Delta, \Theta, \ldots$ (possibly with subscripts) are used to denote such finite sets, and often we write (for example) $\alpha, \beta, \Gamma, \Delta$ for the union $\{\alpha, \beta\} \cup \Gamma \cup \Delta$. In addition, if Γ is the set $\{\alpha_1, \ldots, \alpha_m\}$ then $\neg\Gamma$ and $\neg\mathsf{C}(\Gamma)$ stand for the sets $\{\neg\alpha_1, \ldots, \neg\alpha_m\}$ and $\{\neg\mathsf{C}(\alpha_1), \ldots, \neg\mathsf{C}(\alpha_m)\}$, respectively.

Axioms of $\overline{\mathbf{K}}_n(\mathsf{C})$

(ID) $\qquad P, \neg P, \Gamma$

Basic rules of inference of $\overline{\mathbf{K}}_n(\mathsf{C})$

$(\vee)$
$$\frac{\alpha,\ \beta,\ \Gamma}{\alpha \vee \beta,\ \Gamma}$$

$(\wedge)$
$$\frac{\alpha,\ \Gamma \qquad \beta,\ \Gamma}{\alpha \wedge \beta,\ \Gamma}$$

(K_i)
$$\frac{\alpha,\ \neg\Gamma,\ \neg\mathsf{C}(\Delta)}{\mathsf{K}_i(\alpha),\ \neg\mathsf{K}_i(\Gamma),\ \neg\mathsf{C}(\Delta),\ \Theta}$$

C-rules of $\overline{\mathbf{K}}_n(\mathsf{C})$

$(\neg\mathsf{C})$
$$\frac{\neg\mathsf{E}(\alpha),\ \Gamma}{\neg\mathsf{C}(\alpha),\ \Gamma}$$

$(\mathsf{C}.1)$
$$\frac{\mathsf{E}(\alpha),\ \neg\mathsf{C}(\Delta)}{\mathsf{C}(\alpha),\ \neg\mathsf{C}(\Delta),\ \Theta}$$

$(\mathsf{C}.2)$
$$\frac{\neg\mathsf{E}(\alpha),\ \mathsf{E}^2(\alpha),\ \neg\mathsf{C}(\Delta)}{\neg\mathsf{E}(\alpha),\ \mathsf{C}(\alpha),\ \neg\mathsf{C}(\Delta),\ \Theta}$$

We will see later that the rules $(\mathsf{C}.1)$ and $(\mathsf{C}.2)$ are the counterparts of the induction rule (IND) of $\mathbf{K}_n(\mathsf{C})$. The axioms and rules of our Tait-style formalization of $\mathbf{K}_n(\mathsf{C})$ do not comprise cuts; since we want to distinguish between various cut rules, we always mention explicitly what sort of cuts we use.

Now we introduce the usual cuts, called general cuts in our present context, and two special forms of the cut rule. Other restrictions of the cut rule will be discussed later.

General cuts

(G-Cut)
$$\frac{\alpha,\ \Gamma \qquad \neg\alpha,\ \Gamma}{\Gamma}$$

The designated formulas α and $\neg\alpha$ are called the *cut formulas* of this general cut.

Special cuts

(S1-Cut)
$$\frac{\mathsf{C}(\alpha),\ \neg\mathsf{E}(\alpha),\ \Gamma \qquad \neg\mathsf{C}(\alpha),\ \neg\mathsf{E}(\alpha),\ \Gamma}{\neg\mathsf{E}(\alpha),\ \Gamma}$$

(S2-Cut)
$$\frac{\mathsf{E}(\alpha),\ \mathsf{C}(\alpha),\ \Gamma \qquad \neg\mathsf{E}(\alpha),\ \mathsf{C}(\alpha),\ \Gamma}{\mathsf{C}(\alpha),\ \Gamma}$$

The designated formulas $\mathsf{C}(\alpha)$ and $\neg\mathsf{C}(\alpha)$ for (S1-Cut) and the designated formulas $\mathsf{E}(\alpha)$ and $\neg\mathsf{E}(\alpha)$ for (S2-Cut) are the respective *cut formulas* of these special cuts.

Derivability of a finite set Γ of $L_n(\mathsf{C})$ formulas in $\overline{\mathbf{K}}_n(\mathsf{C})$ with possible additional cuts from ($*_1$-Cut), ..., ($*_m$-Cut) is introduced as usual and written as

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (*_1\text{-Cut}) + \ldots + (*_m\text{-Cut}) \vdash \Gamma.$$

Before saying more about general and special cuts, we have to make sure that $\overline{\mathbf{K}}_n(\mathsf{C})+$(G-Cut) is a reformulation of $\mathbf{K}_n(\mathsf{C})$. One direction is straightforward and formulated below.

Lemma 2 *We have for all $L_n(\mathsf{C})$ formulas $\alpha_1, \ldots, \alpha_m$ that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \alpha_1, \ldots, \alpha_m \quad \Longrightarrow \quad \mathbf{K}_n(\mathsf{C}) \vdash \alpha_1 \vee \ldots \vee \alpha_m.$$

The proof of this lemma is unproblematic but requires some tedious work within the theory $\mathbf{K}_n(\mathsf{C})$ which we omit. For establishing the reduction of $\mathbf{K}_n(\mathsf{C})$ to $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut), it is convenient to begin with some auxiliary considerations. A first remark refers to the propositional completeness and the co-closure properties of $\overline{\mathbf{K}}_n(\mathsf{C})$.

Lemma 3 *For all $L_n(\mathsf{C})$ formulas α the following two assertions can be proved in $\overline{\mathbf{K}}_n(\mathsf{C})$:*

1. $\neg\alpha,\ \alpha$.
2. $\neg\mathsf{C}(\alpha),\ \mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha))$.

Proof The first assertion can be easily established by induction on the complexity measure $\mathsf{me}(\alpha)$ of α; details are left to the reader. Thus we have

$$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{E}(\alpha),\ \mathsf{E}(\alpha), \tag{1}$$

$$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{C}(\alpha),\ \mathsf{C}(\alpha). \tag{2}$$

From (1) we can immediately deduce by rule ($\neg\mathsf{C}$) that

$$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{C}(\alpha),\ \mathsf{E}(\alpha), \tag{3}$$

Moreover, (2) and applications of the rules $(\mathsf{K}_1), \ldots, (\mathsf{K}_n)$ and $(\wedge)$ yield

$$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{C}(\alpha),\ \mathsf{E}(\mathsf{C}(\alpha)). \tag{4}$$

Altogether, statements (3) and (4) plus once more the rule $(\wedge)$ give us what we want. □

Lemma 4 *For all $L_n(\mathsf{C})$ formulas α and β the following two assertions can be proved in $\overline{\mathbf{K}}_n(\mathsf{C})$:*

1. $\neg\mathsf{C}(\alpha),\ \neg\mathsf{C}(\alpha \to \beta),\ \mathsf{C}(\beta)$.
2. $\neg\mathsf{E}(\alpha),\ \neg\mathsf{C}(\mathsf{E}(\alpha)),\ \mathsf{C}(\alpha)$.

Proof We work informally in the system $\overline{\mathbf{K}}_n(\mathsf{C})$. Then a series of applications of its basic rules yields

$$\neg\mathsf{E}(\alpha),\ \neg\mathsf{E}(\alpha \to \beta),\ \mathsf{E}(\beta). \tag{1}$$

Now we apply the rule $(\neg\mathsf{C})$ two times and obtain

$$\neg\mathsf{C}(\alpha),\ \neg\mathsf{C}(\alpha \to \beta),\ \mathsf{E}(\beta). \tag{2}$$

In the next step we make use of rule $(\mathsf{C}.1)$ in order to derive

$$\neg\mathsf{C}(\alpha),\ \neg\mathsf{C}(\alpha \to \beta),\ \mathsf{C}(\beta) \tag{3}$$

and thus have a proof of the first assertion of our lemma. For the second, observe that

$$\neg\mathsf{E}(\alpha),\ \neg\mathsf{E}^2(\alpha),\ \mathsf{E}^2(\alpha) \tag{4}$$

because of Lemma 3. In view of rule $(\neg\mathsf{C})$ we are therefore also able to prove

$$\neg\mathsf{E}(\alpha),\ \neg\mathsf{C}(\mathsf{E}(\alpha)),\ \mathsf{E}^2(\alpha) \tag{5}$$

and are now in the position to continue with rule $(\mathsf{C}.2)$ which provides the desired result. $\square$

Lemma 5 *Let α and β be two $L_n(\mathsf{C})$ formulas so that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \neg\beta,\ \mathsf{E}(\alpha) \land \mathsf{E}(\beta).$$

Then we also have that

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \neg\beta,\ \mathsf{C}(\alpha).$$

Proof Again we work informally in the system $\overline{\mathbf{K}}_n(\mathsf{C})$. The assumption and some simple transformations yield

$$\neg\beta,\ \mathsf{E}(\alpha) \tag{1}$$

and

(2) $$\neg\beta,\ \mathsf{E}(\beta).$$

From (1) and (2) we obtain by applications of the rules $(\mathsf{K}_1), \ldots, (\mathsf{K}_n)$ and some other basic rules that

(3) $$\mathsf{E}(\beta \to \mathsf{E}(\alpha))$$

and

(4) $$\neg\mathsf{E}(\beta),\ \mathsf{E}^2(\beta).$$

With rule $(\mathsf{C}.1)$ we can transform (3) into

(5) $$\mathsf{C}(\beta \to \mathsf{E}(\alpha)),$$

whereas $(\mathsf{C}.2)$ applied to (4) gives

(6) $$\neg\mathsf{E}(\beta),\ \mathsf{C}(\beta).$$

In view of Lemma 4(1), we can now rewrite (5) as

(7) $$\neg\mathsf{C}(\beta),\ \mathsf{C}(\mathsf{E}(\alpha))).$$

Therefore, by means of two cuts, lines (2), (6) and (7) allow us to conclude that

(8) $$\neg\beta,\ \mathsf{C}(\mathsf{E}(\alpha))).$$

Hence the assertion of our lemma follows with two further cuts from (1) and (8) and Lemma 4(2). □

Theorem 6 *We have for all $L_n(\mathsf{C})$ formulas α that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + \text{(G-Cut)} \vdash \alpha \quad \Longleftrightarrow \quad \mathbf{K}_n(\mathsf{C}) \vdash \alpha.$$

Proof The direction from left to right is a direct consequence of Lemma 2. In order to prove the converse direction, we first observe that the basic axioms of $\mathbf{K}_n(\mathsf{C})$ are trivially derivable in $\overline{\mathbf{K}}_n(\mathsf{C})$ and that the co-closure axioms are proved in Lemma 3(2). Hence all axioms of $\mathbf{K}_n(\mathsf{C})$ are provable in $\overline{\mathbf{K}}_n(\mathsf{C})$. Since Lemma 5 states that $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) is closed under the induction rule of $\mathbf{K}_n(\mathsf{C})$ and since all other derivation rules of $\mathbf{K}_n(\mathsf{C})$ have obvious counterparts in $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut), the direction from right to left of our theorem follows by induction on the derivations in $\mathbf{K}_n(\mathsf{C})$. □

The following theorem is our first result about partial cut elimination for logics of common knowledge: it states that all general cuts can be eliminated on the price of permitting special cuts. Of course, (S1-Cut) and (S1-Cut) have been designed so that this is the case.

Theorem 7 (Partial cut elimination 1) *We have for all finite sets Γ of $L_n(\mathsf{C})$ formulas that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + \text{(G-Cut)} \vdash \Gamma \quad \Longrightarrow \quad \overline{\mathbf{K}}_n(\mathsf{C}) + \text{(S1-Cut)} + \text{(S2-Cut)} \vdash \Gamma.$$

This theorem is easily proved by adapting the standard cut elimination procedure for propositional or predicate logic to our present situation.

Proofs in $\overline{\mathbf{K}}_n(\mathsf{C})$ + (S1-Cut) + (S2-Cut) do not have the subformula property, but all cut formulas which may occur in special cuts must have "close relatives" in the corresponding conclusions. Thus the search space for the reconstruction of a proof of a provable formula is reduced significantly.

Complete cut elimination for $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) or more sophisticated Tait- or Gentzen-calculi for $\mathbf{K}_n(\mathsf{C})$ seems not to be possible. Let us work with two agents only, choose two different atomic propositions P and Q and consider the formula α given by

$$\neg\mathsf{K}_1(P \land \mathsf{C}(Q)) \;\lor\; \neg\mathsf{K}_2(Q \land \mathsf{C}(P)) \;\lor\; \mathsf{C}(P \lor Q).$$

Then it can be easily checked that $\mathbf{K}_n(\mathsf{C}) \models \alpha$, hence $\overline{\mathbf{K}}_n(\mathsf{C}) + \text{(G-Cut)} \vdash \alpha$ in view of Theorem 1 and Theorem 6. But it is also not too complicated to show that α cannot be proved in (the cut-free system) $\overline{\mathbf{K}}_n(\mathsf{C})$.

4 Fischer-Ladner cuts

A further interesting partial cut elimination result for a Tait-style version of $\mathbf{K}_n(\mathsf{C})$ is obtained by restricting cuts to specific formulas generated from the Fischer-Ladner closure of provable formulas. The exact details will be described below; first we introduce a variant $\widehat{\mathbf{K}}_n(\mathsf{C})$ of $\overline{\mathbf{K}}_n(\mathsf{C})$ which is more convenient for our purposes.

The axioms and basic rules of inference of $\widehat{\mathbf{K}}_n(\mathsf{C})$ are exactly as in $\overline{\mathbf{K}}_n(\mathsf{C})$; also the rules $(\neg\mathsf{C})$ and $(\mathsf{C}.1)$ are transferred from $\overline{\mathbf{K}}_n(\mathsf{C})$ to $\widehat{\mathbf{K}}_n(\mathsf{C})$. An inspection of the proof of Lemma 3 thus makes it clear that it is also available for $\widehat{\mathbf{K}}_n(\mathsf{C})$.

Lemma 8 *For all $L_n(\mathsf{C})$ formulas α the following two assertions can be proved in $\widehat{\mathbf{K}}_n(\mathsf{C})$:*

1. $\neg\alpha,\ \alpha$.

2. $\neg\mathsf{C}(\alpha),\ \mathsf{E}(\alpha) \land \mathsf{E}(\mathsf{C}(\alpha))$.

The only difference between $\overline{\mathbf{K}}_n(\mathsf{C})$ and $\widehat{\mathbf{K}}_n(\mathsf{C})$ pertains to the rules (C.2), which have been used to handle the induction rules (IND) of $\mathbf{K}_n(\mathsf{C})$ within $\overline{\mathbf{K}}_n(\mathsf{C})$. They are replaced in $\widehat{\mathbf{K}}_n(\mathsf{C})$ by "proper" induction rules.

Induction rules of $\widehat{\mathbf{K}}_n(\mathsf{C})$

(Ind) $$\frac{\neg\alpha,\ \mathsf{E}(\alpha),\ \neg\mathsf{C}(\Delta) \qquad \neg\alpha,\ \mathsf{E}(\beta),\ \neg\mathsf{C}(\Delta)}{\neg\alpha,\ \mathsf{C}(\beta),\ \neg\mathsf{C}(\Delta),\ \Theta}$$

The rule (Ind) comprises (C.2) as a special case. Since it is also clear that (the obvious translation of) (Ind) can be dealt with in $\mathbf{K}_n(\mathsf{C})$, we obtain the analogue of Theorem 6.

Theorem 9 *We have for all $L_n(\mathsf{C})$ formulas α that*

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + \text{(G-Cut)} \vdash \alpha \quad \Longleftrightarrow \quad \mathbf{K}_n(\mathsf{C}) \vdash \alpha.$$

Hence $\widehat{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) is a second reformulation of $\mathbf{K}_n(\mathsf{C})$ – in fact only a minor modification of $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut). Our next aim is to replace the general cuts (G-Cut) by cuts which can be controlled by the endformula of a proof in this system.

Let Π be a set of $L_n(\mathsf{C})$ formulas which is closed under negation; i.e. Π has the property that $\Pi = \neg\Pi$. Then the Π-*cuts* are all cuts

(Π-Cut) $$\frac{\alpha,\ \Gamma \qquad \neg\alpha,\ \Gamma}{\Gamma}$$

so that their cut formulas α and $\neg\alpha$ belong to the set Π. Such Π-cuts, for very specific sets of formulas Π, will play an important role later.

Lemma 10 *Let Π be a set of $L_n(\mathsf{C})$ formulas which is closed under negation. Then we have for all (finite) sets Γ of $L_n(\mathsf{C})$ formulas and all formulas $(\alpha \vee \beta)$ and $(\alpha_0 \wedge \alpha_1)$ which belong to Π:*

1. $\widehat{\mathbf{K}}_n(\mathsf{C}) + (\Pi\text{-Cut}) \vdash (\alpha \vee \beta),\ \Gamma \quad \Longrightarrow \quad \widehat{\mathbf{K}}_n(\mathsf{C}) + (\Pi\text{-Cut}) \vdash \alpha,\ \beta,\ \Gamma.$
2. $\widehat{\mathbf{K}}_n(\mathsf{C}) + (\Pi\text{-Cut}) \vdash (\alpha_0 \wedge \alpha_1),\ \Gamma \quad \Longrightarrow \quad \widehat{\mathbf{K}}_n(\mathsf{C}) + (\Pi\text{-Cut}) \vdash \alpha_i,\ \Gamma.$

PROOF Obvious derivations in $\widehat{\mathbf{K}}_n(\mathsf{C})$ yield that

(1) $$\widehat{\mathbf{K}}_n(\mathsf{C}) \vdash (\neg\alpha \wedge \neg\beta),\ \alpha,\ \beta,$$

(2) $$\widehat{\mathbf{K}}_n(\mathsf{C}) \vdash (\neg\alpha_0 \vee \neg\alpha_1),\ \alpha_i$$

for $i = 0, 1$. Since the formulas $(\neg\alpha \land \neg\beta)$ and $(\neg\alpha_0 \lor \neg\alpha_1)$ belong to Π, the assertions of our lemma follow from (1) and (2) by simple Π-cuts. $\square$

Let Π and Σ be two sets of $L_n(\mathsf{C})$ formulas which are closed under negation and assume that Σ is a subset of Π. For those Π and Σ we introduce as auxiliary notation:

- A (finite) subset Γ of Π is called Π-*consistent* in case that
$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\Pi\text{-Cut}) \nvdash \neg\Gamma.$$
- A (finite) subset Γ of Σ is called *maximal* Π-*consistent with respect to* Σ if Γ is Π-consistent and if there exists no Π-consistent subset of Σ which is a proper superset of Γ.

Some important properties of maximal Π-consistent sets with respect to Σ are summarized in the subsequent lemma. Its proof is standard and can be omitted.

Lemma 11 *Let Π and Σ be sets of $L_n(\mathsf{C})$ formulas as above. Then we have for all (finite) subsets Γ of Σ, which are maximal Π-consistent with respect to Σ, and all $L_n(\mathsf{C})$ formulas α and β:*

1. $\alpha \in \Sigma \quad \Longrightarrow \quad \alpha \in \Gamma$ *or* $\neg\alpha \in \Gamma$.
2. $\alpha \in \Sigma$ *and* $\widehat{\mathbf{K}}_n(\mathsf{C}) + (\Pi\text{-Cut}) \vdash \neg\Gamma, \alpha \quad \Longrightarrow \quad \alpha \in \Gamma$.
3. $\alpha, \beta \in \Sigma$ *and* $(\alpha \lor \beta) \in \Gamma \quad \Longrightarrow \quad \alpha \in \Gamma$ *or* $\beta \in \Gamma$.
4. $\alpha, \beta \in \Sigma$ *and* $(\alpha \land \beta) \in \Gamma \quad \Longrightarrow \quad \alpha \in \Gamma$ *and* $\beta \in \Gamma$.

Again, let Π and Σ be sets of $L_n(\mathsf{C})$ formulas as above. Each finite set Γ of $L_n(\mathsf{C})$ formulas which is Π-consistent with respect to Σ can be easily extended to a maximal Π-consistent set with respect to Σ. For example, simply fix an enumeration $\gamma_0, \gamma_1, \ldots$ of Σ and define $\Gamma_0 := \Gamma$ as well as

$$\Gamma_{i+1} \ := \ \begin{cases} \Gamma_i \cup \{\gamma_i\} & \text{if } \Gamma_i \cup \{\gamma_i\} \text{ is } \Pi\text{-consistent with respect to } \Sigma, \\ \Gamma_i \cup \{\neg\gamma_i\} & \text{otherwise.} \end{cases}$$

Then simple induction on i shows that each Γ_i is Π-consistent with respect to Σ. Hence the union of all sets Γ_i is a possible candidate for the set Δ which is claimed to exist in the following lemma.

Lemma 12 *Let Π and Σ be sets of $L_n(\mathsf{C})$ formulas as above and assume that Γ is Π-consistent with respect to Σ. Then there exists a subset Δ of Σ which is maximal Π-consistent with respect to Σ and contains Γ.*

Before formulating and proving the main result of this section, we have to fix the sets of $L_n(\mathsf{C})$ formulas which are central for what we want. The so-called *Fischer-Ladner closure* $\mathbb{FL}(\alpha)$ of an $L_n(\mathsf{C})$ formula α (see Fischer and Ladner [4]) is the set of $L_n(\mathsf{C})$ formulas which is inductively generated as follows:

1. α belongs to $\mathbb{FL}(\alpha)$.
2. If β belongs to $\mathbb{FL}(\alpha)$, then $\neg\beta$ belongs to $\mathbb{FL}(\alpha)$.
3. If $(\beta \vee \gamma)$ belongs to $\mathbb{FL}(\alpha)$, then β and γ belong to $\mathbb{FL}(\alpha)$.
4. If $(\beta \wedge \gamma)$ belongs to $\mathbb{FL}(\alpha)$, then β and γ belong to $\mathbb{FL}(\alpha)$.
5. If $\mathsf{K}_i(\beta)$ belongs to $\mathbb{FL}(\alpha)$, then β belongs to $\mathbb{FL}(\alpha)$.
6. If $\mathsf{C}(\beta)$ belongs to $\mathbb{FL}(\alpha)$, then β, $\mathsf{E}(\beta)$ and $\mathsf{E}(\mathsf{C}(\beta))$ belong to $\mathbb{FL}(\alpha)$.

According to [4], the number of elements of $\mathbb{FL}(\alpha)$ is of order $\mathcal{O}(|\alpha|)$ where $|\alpha|$ denotes the length of the formula α.

The set $\mathbb{FL}(\alpha)$ will take over the role of the set Σ in the previous considerations; the counterpart of the set Π will be the *disjunctive-conjunctive closure* of $\mathbb{FL}(\alpha)$ which will be carefully introduced now.

Given an $L_n(\mathsf{C})$ formula α, we fix arbitrary enumerations

$$(\star) \qquad \delta_1, \delta_2, \ldots, \delta_p \quad \text{and} \quad \Delta_1, \Delta_2, \ldots, \Delta_q \quad \text{and} \quad D_1, D_2, \ldots, D_r$$

of the elements of $\mathbb{FL}(\alpha)$, the subsets of $\mathbb{FL}(\alpha)$ and the subsets of the power set $Pow(\mathbb{FL}(\alpha))$ of $\mathbb{FL}(\alpha)$, respectively. Each $\Delta \subset \mathbb{FL}(\alpha)$ can then be written as

$$\{\, \delta_{s(1)}, \delta_{s(2)}, \ldots, \delta_{s(m_\Delta)} \,\}$$

so that $1 \leq s(1) < s(2) < \ldots < s(m_\Delta) \leq p$, and we define

$$(\star\star) \qquad \varphi_\Delta \;:=\; (\ldots(\delta_{s(1)} \wedge \delta_{s(2)}) \wedge \ldots) \wedge \delta_{s(m_\Delta)}).$$

In addition, each $D \subset Pow(\mathbb{FL}(\alpha))$ can be brought into the form

$$\{\, \Delta_{t(1)}, \Delta_{t(2)}, \ldots, \Delta_{t(m_D)} \,\}$$

so that $1 \leq t(1) < t(2) < \ldots < t(m_D) \leq q$, and now we define

$$\varphi_D \;:=\; (\ldots(\varphi_{\Delta_{t(1)}} \vee \varphi_{\Delta_{t(2)}}) \vee \ldots) \vee \varphi_{\Delta_{t(m_D)}}).$$

Finally, we let $\mathbb{DC}(\alpha)$ be the set of all formulas $\varphi_{D_1}, \ldots, \varphi_{D_r}$ and their negations,

$$\mathbb{DC}(\alpha) \;:=\; \{\, \varphi_{D_1}, \ldots, \varphi_{D_r} \,\} \cup \{\, \neg\varphi_{D_1}, \ldots, \neg\varphi_{D_r} \,\}.$$

According to these definitions, $\mathbb{FL}(\alpha)$ is contained in the set $\mathbb{DC}(\alpha)$ which can be regarded as the disjunctive-conjunctive closure of $\mathbb{FL}(\alpha)$ since $\mathbb{DC}(\alpha)$ contains a representative (modulo logical equivalence) for each formula which is built up from the elements of $\mathbb{FL}(\alpha)$ by disjunctions and conjunctions.

Depending on α, we can introduce the *canonical Kripke-frame*

$$\mathfrak{M}^\alpha \;:=\; (W^\alpha, \mathbb{K}_1^\alpha, \ldots, \mathbb{K}_n^\alpha)$$

whose set of worlds W^α is the collection of all maximal $\mathbb{DC}(\alpha)$-consistent sets with respect to $\mathbb{FL}(\alpha)$; the accessibility relations $\mathbb{K}_i^\alpha$ consist of all pairs (Γ, Δ) of elements of W^α so that

$$\Gamma/\mathsf{K}_i \;:=\; \{\beta : \mathsf{K}_i(\beta) \in \Gamma\}$$

is contained in Δ, i.e.

$$\mathbb{K}_i^\alpha \;:=\; \{\, (\Gamma, \Delta) \in W^\alpha \times W^\alpha : \Gamma/\mathsf{K}_i \subset \Delta \,\}.$$

Lemma 13 *Assume that* $\Gamma \in W^\alpha$*,* $(\Gamma, \Delta) \in \mathbb{K}_i^\alpha$ *and* $\mathsf{C}(\gamma) \in \Gamma$*. Then we have* $\mathsf{C}(\gamma) \in \Delta$ *and* $\gamma \in \Delta$*.*

PROOF Recall from Lemma 8(2) that

(1) $$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \;\vdash\; \neg\mathsf{C}(\gamma),\; \mathsf{E}(\mathsf{C}(\gamma)),$$

(2) $$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \;\vdash\; \neg\mathsf{C}(\gamma),\; \mathsf{E}(\gamma).$$

Since $\mathsf{E}(\mathsf{C}(\gamma))$ and all $\mathsf{E}(\gamma)$ belong to $\mathbb{FL}(\alpha)$, we are in the position of applying Lemma 11(2) to (1) and (2) and know that

(3) $$\mathsf{E}(\mathsf{C}(\gamma)) \in \Gamma,$$

(4) $$\mathsf{E}(\gamma) \in \Gamma.$$

Because of Lemma 11(4) we thus have $\mathsf{K}_i(\mathsf{C}(\gamma)) \in \Gamma$ and $\mathsf{K}_i(\gamma) \in \Gamma$. The definition of $\mathbb{K}_i^\alpha$ therefore implies the assertion of our lemma. □

As *canonical valuation* we fix the mapping $\mathcal{V}^\alpha$ from the atomic propositions to $Pow(W^\alpha)$ given by

$$\mathcal{V}^\alpha(P) \;:=\; \{\Gamma \in W^\alpha : P \in \Gamma\}$$

for all elements P of PROP. With $\mathfrak{M}^\alpha$ and $\mathcal{V}^\alpha$ being provided, we are ready for establishing the main lemma for the proof of Theorem 15.

Lemma 14 *Let α be an $L_n(\mathsf{C})$ formula. Then we have for all $\Gamma \in W^\alpha$ and all $\beta \in \mathbb{FL}(\alpha)$ that*

$$\beta \in \Gamma \quad \Longleftrightarrow \quad (\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Gamma) \models \beta.$$

Proof We show this equivalence by induction on the structure of the formula β and carry through the following distinction by cases.

1. β is an atomic proposition or the negation of an atomic proposition. Then the assertion follows from the definition of $\mathcal{V}^\alpha$.

2. β is of the form $(\gamma_0 \vee \gamma_1)$ or $(\gamma_0 \wedge \gamma_1)$. Then the assertion follows from the induction hypothesis by means of Lemma 11.

3. β is of the form $\mathsf{K}_i(\gamma)$. The direction from left to right is immediate from the definition of $\mathbb{K}_i^\alpha$ and the induction hypothesis. For the converse direction, assume that $\mathsf{K}_i(\gamma) \notin \Gamma$. Then $\neg\mathsf{K}_i(\gamma) \in \Gamma$ by Lemma 11(1) and

$$(1) \qquad \widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \not\vdash \mathsf{K}_i(\gamma),\ \{\neg\mathsf{K}_i(\delta) : \mathsf{K}_i(\delta) \in \Gamma\}.$$

Because of the rule (K_i) we therefore also have

$$(2) \qquad \widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \not\vdash \gamma,\ \{\neg\delta : \mathsf{K}_i(\delta) \in \Gamma\}.$$

This means that the set $\{\neg\gamma\} \cup \{\delta : \mathsf{K}_i(\delta) \in \Gamma\}$ is $\mathbb{DC}(\alpha)$-consistent with respect to $\mathbb{FL}(\alpha)$. Note that Lemma 12 therefore claims the existence of an element Δ of W^α with

$$(3) \qquad \neg\gamma \in \Delta,$$

$$(4) \qquad \{\delta : \mathsf{K}_i(\delta) \in \Gamma\} \subset \Delta.$$

From (3) we conclude with the induction hypothesis that $(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Delta) \not\models \gamma$. Further, (4) yields that $(\Gamma, \Delta) \in \mathbb{K}_i^\alpha$. Hence $(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Gamma) \not\models \mathsf{K}_i(\gamma)$, and the direction from right to left is proved.

4. β is of the form $\neg\mathsf{K}_i(\gamma)$. The treatment of this case is analogous to the previous one.

5. β is of the form $\mathsf{C}(\gamma)$. For showing the direction from left to right we assume $\mathsf{C}(\gamma) \in \Gamma$. Lemma 13 and a simple proof by induction on m entails that

$$(5) \qquad \mathsf{C}(\gamma) \in \Delta \quad \text{and} \quad \gamma \in \Delta$$

for all elements $\Delta \in W^\alpha$ which are accessible from Γ in m steps. But then the induction hypothesis implies

$$(6) \qquad (\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Delta) \models \gamma$$

for such Δ. Given the definition of the validity of the formula $\mathsf{C}(\gamma)$, we have herewith shown that $(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Gamma) \models \mathsf{C}(\gamma)$.

For dealing with the converse direction, we first recall the enumerations $(\star)$ and let

$$\Delta_{w(1)},\ \Delta_{w(2)},\ \ldots,\ \Delta_{w(u)}$$

with $1 \leq w(1) < w(2) < \ldots < w(u) \leq q$ be the list of all sets Δ_j so that $(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Delta_j) \models \mathsf{C}(\gamma)$. Now introduce the formula $\psi_{\mathsf{C}(\gamma)}$,

$$\psi_{\mathsf{C}(\gamma)} \ := \ (\ldots(\varphi_{\Delta_{w(1)}} \vee \varphi_{\Delta_{w(2)}}) \vee \ldots) \vee \varphi_{\Delta_{w(2)}})$$

for each $\varphi_{\Delta_{w(j)}}$ being defined as in $(\star\star)$. From the definition of $\mathbb{DC}(\alpha)$ above we learn that $\psi_{\mathsf{C}(\gamma)} \in \mathbb{DC}(\alpha)$. For this formula $\psi_{\mathsf{C}(\gamma)}$ we want to show:

$$\widehat{\mathbf{K}}_n(\mathsf{C}) \ \vdash\ \neg\psi_{\mathsf{C}(\gamma)},\ \mathsf{E}(\gamma), \tag{7}$$

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash\ \neg\psi_{\mathsf{C}(\gamma)},\ \mathsf{E}(\psi_{\mathsf{C}(\gamma)}). \tag{8}$$

To prove (7), observe that

$$(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Delta) \models \mathsf{C}(\gamma) \quad\Longrightarrow\quad (\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Delta) \models \mathsf{E}(\gamma)$$

for all $\Delta \in W^\alpha$. Hence the induction hypothesis tells us that $\mathsf{E}(\gamma) \in \Delta_{w(j)}$ for $j = 1, \ldots, u$. Consequently, we have

$$\widehat{\mathbf{K}}_n(\mathsf{C}) \ \vdash\ \neg\varphi_{\Delta_{w(j)}},\ \mathsf{E}(\gamma) \tag{9}$$

for $j = 1, \ldots, u$. From (9) and the definition of $\neg\psi_{\mathsf{C}(\gamma)}$ we obtain assertion (7) by some obvious basic inferences.

The proof of (8) is more complicated: We first observe that for all $\Delta \in W^\alpha$

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash\ \neg(\Delta/\mathsf{K}_i),\ \{\varphi_{\Delta_j} : j \in N_\Delta\} \tag{10}$$

where N_Δ is the set of all natural numbers given by

$$N_\Delta \ := \ \{\, j : 1 \leq j \leq q \ \text{ and } \ (\Delta, \Delta_j) \in \mathbb{K}_i^\alpha \,\},$$

again referring to the enumerations $(\star)$. If this were not the case, then we could pick for each $j \in N_\Delta$ a formula $\chi_j \in \Delta_j$ satisfying

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \nvdash\ \neg(\Delta/\mathsf{K}_i),\ \{\chi_j : j \in N_\Delta\}.$$

However, this would imply that the set

$$(\Delta/\mathsf{K}_i) \ \cup\ \{\neg\chi_j : j \in N_\Delta\}$$

is $\mathbb{DC}(\alpha)$-consistent with respect to $\mathbb{FL}(\alpha)$ and therefore, by Lemma 12, contained in a set Θ which is maximal $\mathbb{DC}(\alpha)$-consistent with respect to $\mathbb{FL}(\alpha)$. But then we had $(\Delta/\mathsf{K}_i) \subset \Theta$, hence $(\Delta, \Theta) \in \mathbb{K}_i^\alpha$, and $\Theta \neq \Delta_j$ for all $j \in N_\Delta$ because of the choice of the formulas χ_j. This is a contradiction, and (10) has been established.

The next step is to choose an arbitrary $\Delta_{w(k)}$ with $1 \leq k \leq u$. By (10) we have

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg(\Delta_{w(k)}/\mathsf{K}_i),\ \psi_{\mathsf{C}(\gamma)}, \tag{11}$$

simply because $N_{\Delta_{w(k)}} \subset \{w(1), w(2), \ldots, w(u)\}$. By applying the rule (K_i) to (11) we gain

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\Delta_{w(k)},\ \mathsf{K}_i(\psi_{\mathsf{C}(\gamma)}), \tag{12}$$

hence also

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\Delta_{w(k)},\ \mathsf{E}(\psi_{\mathsf{C}(\gamma)}), \tag{13}$$

since (12) holds for all operators $\mathsf{K}_1, \ldots, \mathsf{K}_n$. Assertion (13) is immediately transformed into

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\varphi_{\Delta_{w(k)}},\ \mathsf{E}(\psi_{\mathsf{C}(\gamma)}) \tag{14}$$

and available for all $1 \leq k \leq u$. Therefore assertion (8) follows from (14) by several applications of the rule $(\wedge)$.

Having proved assertions (7) and (8), the induction rule (Ind) comes into play and yields

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\psi_{\mathsf{C}(\gamma)},\ \mathsf{C}(\gamma). \tag{15}$$

Since $\psi_{\mathsf{C}(\gamma)}$ belongs to $\mathbb{DC}(\alpha)$, assertion (15) gives us in view of Lemma 10(2) that

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\varphi_{\Delta_{w(k)}},\ \mathsf{C}(\gamma) \tag{16}$$

for all $1 \leq k \leq u$. The formulas $\varphi_{\Delta_{w(k)}}$ are elements of $\mathbb{DC}(\alpha)$ as well, and now we apply Lemma 10(1) to (16) in order to obtain

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\Delta_{w(k)},\ \mathsf{C}(\gamma) \tag{17}$$

for all $1 \leq k \leq u$. To conclude the proof of the direction from right to left, assume that $\Gamma \in W^\alpha$ and

$$(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Gamma) \models \mathsf{C}(\gamma).$$

Then the set Γ is identical to some $\Delta_{w(k)}$, $1 \leq k \leq u$, and thus (17) entails

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \neg\Gamma,\ \mathsf{C}(\gamma). \tag{18}$$

Finally we make use of Lemma 11(2) and gain $\mathsf{C}(\gamma) \in \Gamma$, as desired.

6. β is of the form $\neg\mathsf{C}(\gamma)$. The treatment of this case is analogous to the previous one. □

Theorem 15 *For all $L_n(\mathsf{C})$ formulas α we have that*

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \alpha \quad \Longleftrightarrow \quad \mathbf{K}_n(\mathsf{C}) \models \alpha.$$

Proof The direction from left to right of this equivalence is implied by Theorem 9 and Theorem 1. Conversely, fix an $L_n(\mathsf{C})$ formula α and assume that

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \nvdash \ \alpha.$$

Then the set $\{\neg\alpha\}$ is $\mathbb{DC}(\alpha)$-consistent with respect to $\mathbb{FL}(\alpha)$, and hence, because of Lemma 12, there must be a set Γ which contains $\neg\alpha$ and is maximal $\mathbb{DC}(\alpha)$-consistent with respect to $\mathbb{FL}(\alpha)$, i.e.

$$\Gamma \in W^\alpha \quad \text{and} \quad \neg\alpha \in \Gamma.$$

Now we can apply the previous lemma in order to obtain

$$(\mathfrak{M}^\alpha, \mathcal{V}^\alpha, \Gamma) \models \neg\alpha.$$

This means that α is not valid in the canonical $\mathfrak{M}^\alpha$, and, consequently, $\mathbf{K}_n(\mathsf{C}) \not\models \alpha$. This completes the proof of our theorem. □

Corollary 16 (Partial cut elimination 2) *For all $L_n(\mathsf{C})$ formulas α we have that*

$$\widehat{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \ \vdash \ \alpha \quad \Longleftrightarrow \quad \widehat{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\alpha)\text{-Cut}) \ \vdash \ \alpha.$$

The last assertion is a trivial consequence of Theorem 1, Theorem 9 and Theorem 15 just above. It says that for each proof of a formula α in $\widehat{\mathbf{K}}_n(\mathsf{C})+$ (G-Cut) there exists a proof with cuts so that all their cut formulas belong to the representation system $\mathbb{DC}(\alpha)$ of the disjunctive-conjunctive closure of the Fischer-Ladner closure of α.

In a subsequent publication we will show that Theorem 15 and Corollary 16 can be improved significantly: only a set much smaller than $\mathbb{DC}(\alpha)$ is needed to achieve a corresponding completeness result and a corresponding result about partial cut elimination.

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