Realization of Constructive Set Theory into Explicit Mathematics: a lower bound for impredicative Mahlo universe

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Abstract

We define a realizability interpretation of Aczel's Constructive Set Theory **CZF** into Explicit Mathematics. The final results are that **CZF** extended by Mahlo principles is realizable in corresponding extensions of $\mathbf{T_0}$, thus providing relative lower bounds for the proof-theoretic strength of the latter.

Introduction

Several different frameworks have been founded in the 70-es aiming to give a foundation for constructive mathematics. The most well-developed of them nowadays are Martin-Löf type theory, Aczel's constructive set theory, and Feferman's explicit mathematics. While constructive set theory was built to have an immediate type interpretation, no theory stronger than $\mathbf{\Delta^1_2}-\mathbf{CA}$, which proof-theoretically is still far below the basic system $\mathbf{T_0}$ of Explicit Mathematics, have been shown up to now to be directly embeddable into explicit systems. It also yielded that the only method for establishing lower bounds for $\mathbf{T_0}$ and its extensions remained to be well-ordering proofs. This omission became apparent again, when Jäger and Studer [JStu] introduced a theory $\mathsf{T_0(M)}$ extending $\mathbf{T_0}$ by a Mahlo axiom and built its model in Kripke-Platek Set Theory **KPM**, but the question of lower bound was left open.
The situation is quite different in Martin-Löf type theory, where, in addition to well-ordering proofs (see [Se98]), we also have direct embeddings of Constructive Set Theory **CZF**, [Acz78], and its extensions, [Acz86, RaCZFM], or a subsystem of analysis **IARI**, [GR94]. These kinds of embeddings into **ML** type theory are often referred to as *realizability interpretations*. The name is justified in the sense that in the type theory logical operations are introduced as shortcuts for certain constructions, and in fact exactly those ones which are assigned to the operations by Kleene realizability, if one takes intuitionistic logic as primitive. This is exactly the way taken by Explicit Mathematics: logic comes first. Then it turns out that it doesn't matter much which logic, intuitionistic or classical, to assume: Explicit Mathematics has proven to have a lot of classical applications, incompatible with intuitionistic point of view. For this reason, even intuitionistically, *realizability* and *derivability* are different phenomena in Explicit Mathematics: there are simple realizable, but not derivable, formulas. It's important however that formal realizability can be elegantly expressed in Explicit Mathematics and is equivalent to derivability for a wide class of formulas, including those expressing proof-theoretic strength[1].
In the present paper we develop a *realizability interpretation of Constructive Set Theory* **CZF** *into Explicit Mathematics*, with a specific purpose of giving lower bounds for Mahlo axioms in the context of the latter. However, our interpretation is applicable to both weaker and stronger variants of **CZF**, as well as in non-Set-Theory setting (see [Tura]).

*In the Russian language the word "malo" means "a little", "not enough"
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[1] See [Tura], Sections 3 and 4, for more about this

In classical set-theoretic terms, an admissible ordinal α is called *recursively Mahlo*, if for each α-recursive function f there exists an admissible ordinal $\beta < \alpha$, which is closed under f. Recently Mahlo-style properties have been studied extensively in different theoretic frameworks. Starting with the works [Ra90, Ra91, Ra94a] by M. Rathjen, who pioneered in recognising importance of mahloness for proof theory, this kind of axioms was introduced into **ML** type theory, [SeM, Ra00], and Constructive Set Theory, [RaCZFM]. In the context of Explicit Mathematics, a very natural Mahlo axiom was proposed in [JStu]. It provides for a uniform operation m for passing from a pair (α, f), in which α is a name and f takes names to names, to $\mathsf{m}(\alpha, f)$, which names a universe containing α and closed under f. The effect of this axiom in *metapredicative* setting, i.e. – in Explicit Mathematics – in theories deprived of inductive generation, has been studied in [JSumM, StrwmM]. We should also mention that many recent developments in higher ordinal analysis, starting with [Ra94b], use the idea of *hyper-mahloness* as one of the basic building blocks.
We consider two versions of Mahlo axioms, both in Constructive Set Theory and in Explicit Mathematics: a weaker one saying that the whole world of objects is Mahlo, and a stronger one claiming existence of a Mahlo object (*set* in **CZF** and *universe* in **EM**). The first one corresponds in strength to the Set Theory **KPM**, constructive counterpart of which is a certain theory **CZFM**, Definition 4.6. In **EM** this idea is caught by *Mahlo operation* m introduced in [JStu]; in the present paper it's described by axioms VII–VIII and the corresponding theory is called $\mathbf{T_m}$. The second form of mahloness corresponds to a certain minor extension $\mathbf{KPM}^+$ of **KPM**, and, constructively, to a theory $\mathbf{CZFM}^+$, Definition 4.9, based on [RaCZFM], and Setzer's Mahlo universe [SeM] in Martin-Löf type theory. In Explicit Mathematics for this purpose we introduce a constant M for a name of *Mahlo universe*, adding which, together with axiom IX, results in a theory $\mathbf{T_M}$. Axiom VII alone, responsible for a theory $\mathbf{T_u}$, is the *limit* axiom of [JStu]; its analogue in Constructive Set Theory is **REA**, and in Kripke-Platek Set Theory a familiar limit axiom in the classical set-theoretic context.
Final results of the paper are the following:

Theorem 4 **CZFM** *is realizable in* $\mathbf{T_m}$

and

Theorem 5 $\mathbf{CZFM}^+$ *is realizable in* $\mathbf{T_M}$.

Since we consider only intuitionistic versions of Explicit Mathematics, these theorems give us the following new knowledge:

$$|\mathbf{CZFM}| \leq |\mathbf{T_m} \text{ intuitionistic}| \leq |\mathbf{T_m} \text{ classical}| \leq |\mathbf{KPM}|$$

and

$$|\mathbf{CZFM}^+| \leq |\mathbf{T_M} \text{ intuitionistic}| \leq |\mathbf{T_M} \text{ classical}| \leq |\mathbf{KPM}^+|,$$

where $|\cdot|$ stands for proof-theoretic strength. Combining this with plausible assumptions

$$|\mathbf{CZFM}| = |\mathbf{KPM}|$$

and

$$|\mathbf{CZFM}^+| = |\mathbf{KPM}^+|,$$

would imply that all the above inequalities are in fact exact and $\mathbf{T_m}$ and $\mathbf{T_M}$ inherit a special for this strength feature of Explicit Mathematics: proof-theoretically the law of excluded middle doesn't matter there.

Now we briefly describe contents of the paper.

After introducing systems of Explicit Mathematics in Section 1 and preparatory work in the beginning of Section 2 we define a notion of (a name of) a *set in Explicit Mathematics* (Definition 2.3), as wellfounded tree consisting of arbitrary objects. Then four kinds of realizability interpretations $\underline{\mathrm{rn}}$, $\underline{\mathrm{rn}}^{\mathcal{E}}$, $\underline{\mathrm{rn}}^{\forall}$ and $\underline{\mathrm{rn}}^{\exists}$ from the language of Set Theory to the language of **EM** are introduced (Definitions 2.6, 2.10 and 2.12,1),2)). The main realizability is $\underline{\mathrm{rn}}$, and in principle it alone would suffice, but other three significantly simplified

some parts of the paper. $\underline{\mathrm{rn}}^{\mathcal{E}}$ is good because it maps bounded formulas into elementary ones. All these realisabilities are not equivalent to each other, but they are *operationally equivalent*, i.e. can be mapped into each other by preset operations (Definitions 2.11 and 2.12,3),4)), which is a usual situation when several kinds of realizability are simultaneously considered. Set equality is realized as *bisimulation* between trees (Definition 2.5); this verifies all equality axioms as well as Extensionality (Theorem 1 and Lemma 3.1).
The switch from higher properties in Set Theory, starting with regularity, to corresponding higher properties in Explicit Mathematics is achieved via key notions of a *universe*, Definition 1.2, and a *universal set*, Definition 4.1. If Reg, In and M are set-theoretic formulas expressing correspondingly *regularity*, *inaccessibility* and *mahloness* of sets (Definitions 4.2, 4.5 and 4.8), then we have the following central lemmas:

Lemma 4.4 *If* υ *names a universe then* $\mathrm{Reg}[\mathsf{us}\upsilon]$ *is realizable in* $\mathbf{T_0}\restriction$,

Lemma 4.6 *If a universe* ι *is inaccessible then* $\mathrm{In}[\mathsf{us}\iota]$ *is realizable in* $\mathbf{T_u}\restriction$,

Lemma 4.7 *Mahlo schema is realizable in* $\mathbf{T_m}\restriction$

and

Lemma 4.8 *Mahlo axiom is realizable in* $\mathbf{T_M}\restriction$.

These lemmas lead us to Theorems 4 and 5.

Our realizability interpretation also applies to theories with restricted induction principles, as shown in the paper: Constructive Set Theory with restricted foundation is realizable in systems of Explicit Mathematics where inductive generation and induction on natural numbers are restricted in a similar way. Set Theories where foundation is omitted altogether were not treated here, but the method should work for them equally well.

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1 Explicit Mathematics. Theories $\mathbf{T_0}$, $\mathbf{T_u}$, $\mathbf{T_m}$ and $\mathbf{T_M}$

We follow essentially the original type-free two-sorted formulation of Explicit Mathematics from [Fef75]. Alternative formulations are given in [Be85] and [Jä88].
Languages $\mathcal{L}_{\mathbf{T_0}}$, $\mathcal{L}_{\mathbf{T_u}}$, $\mathcal{L}_{\mathbf{T_m}}$ and $\mathcal{L}_{\mathbf{T_M}}$. All theories of Explicit Mathematics, considered in this paper, are formulated in a two-sorted language, containing variables for operations (individuals) and names, along with operation constants. Names are thought of as a special kind of operations, coding types (sets) of operations. We use **variables** $a, b, c, \ldots, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ as ranging over operations, and $\alpha, \beta, \gamma, \ldots$ as ranging over names. The **constants** of $\mathcal{L}_{\mathbf{T_0}}$ are the following: combinators k, s, pairing p and projections $\mathsf{p_0}$, $\mathsf{p_1}$, zero 0, successor $\mathsf{s_N}$ and predecessor $\mathsf{p_N}$, distinction by cases on natural numbers $\mathsf{d_N}$, join j and inductive generation i. Additionally we have the following 8 **constants** called *name generators*: nat, id, inv, and, or, imp, all, ex. The languages $\mathcal{L}_{\mathbf{T_u}}$, $\mathcal{L}_{\mathbf{T_m}}$ and $\mathcal{L}_{\mathbf{T_M}}$ are obtained by adjoining to $\mathcal{L}_{\mathbf{T_0}}$ successively a universe constant u, a small Mahlo constant m and a big Mahlo constant M. **Terms** are built from variables and constants by the following application clause: if s and t are *terms* then $s \cdot t$ is a *term*, so that the *application* function symbol $\cdot$ accepts arguments of both sorts and returns an operation. **Atomic formulas** are $s = t$ (s coincides with t) and $s \,\varepsilon\, t$ (s belongs to the set named by t, s is classified under t), where s and t are terms. **Formulas** are built from atomic formulas by $\wedge, \vee, \rightarrow$ and two kinds of quantifiers, over operations and over names, e.g. $\forall a, \exists a, \forall \alpha, \exists \alpha$. Finally, **expression** is a term or a formula.
Syntactical conventions.
1. We use $e[t]$ for an expression e, possibly containing occurrences of an expression t. In this context by $e[s]$ we mean e^s_t, i. e. the result of substituting expression s for all occurrences of t in e.
2. Parentheses in terms are assumed to be associated to the left: e.g., $s \cdot t \cdot u$ is read as $(s \cdot t) \cdot u$.
3. We adopt the following priority among propositional connectives and their abbreviations: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. For example, $F_1 \vee \neg F_2 \wedge F_3 \rightarrow F_4 \leftrightarrow F_5$ has to be read as $((F_1 \vee ((\neg F_2) \wedge F_3)) \rightarrow F_4) \leftrightarrow F_5$.

Abbreviations. We use the following abbreviations:
$\neg F \ :\Leftrightarrow\ F \to 0 = \mathsf{s_N}\cdot 0$;
$F_0 \leftrightarrow F_1 \ :\Leftrightarrow\ (F_0 \to F_1) \wedge (F_1 \to F_0)$;
$t\downarrow \ :\Leftrightarrow\ \exists x(t = x)$;
$\mathcal{N}[t] \ :\Leftrightarrow\ \exists \alpha(t = \alpha)$;
$F[t\downarrow] \ :\Leftrightarrow\ t\downarrow \wedge F[t]$;
$t \doteq \{s[x_1,\ldots,x_n] \mid F[x_1,\ldots,x_n]\} \ :\Leftrightarrow\ \mathcal{N}[t] \wedge \forall x(x\,\varepsilon\, t \leftrightarrow \exists x_1\ldots\exists x_n(x = s[x_1,\ldots,x_n] \wedge F[x_1,\ldots,x_n]))$;
$s \simeq t \ :\Leftrightarrow\ (s\downarrow \vee t\downarrow) \to s = t$;
$s \dot\subseteq t \ :\Leftrightarrow\ \forall x\,\varepsilon\, s(x\,\varepsilon\, t)$; $s \doteq t \ :\Leftrightarrow\ s \dot\subseteq t \wedge t \dot\subseteq s$;
$r : s \mapsto t$ for $\forall x\,\varepsilon\, s(rx\,\varepsilon\, t)$; $r : s^{m+1} \mapsto t$ for $\forall x\,\varepsilon\, s(rx : s^m \mapsto t)$;
t' for $\mathsf{s_N}\cdot t$; 1 for $0'$; 2 for $1'$; st for $s\cdot t$; $t(s_1,\ldots,s_n)$ for $(\ldots(ts_1)\ldots s_n)$; $\langle s,t\rangle$ for $\mathsf{p}st$; $s \neq t$ for $\neg s = t$, etc.
Logic. Intuitionistic 2-sorted logic of partial terms with equality. See, e.g., [Be85] or [Tr98].
Axioms. The axioms are divided in several groups, according to their nature.

I. Applicative axioms. These axioms formalise that operations form a partial combinatory algebra, that we have pairing and projections, usual closure conditions on natural numbers, as well as definition by numerical cases:
(1) $\mathsf{k}ab = a$;
(2) $\mathsf{s}ab\downarrow \wedge\ \mathsf{s}abc \simeq ac(bc)$;
(3) $\mathsf{p}ab\downarrow \wedge\ \mathsf{p_0}a\downarrow \wedge\ \mathsf{p_1}a\downarrow \wedge\ \mathsf{p_0}(\mathsf{p}ab) = a \wedge \mathsf{p_1}(\mathsf{p}ab) = b$;
(4) $0\,\varepsilon\,\mathsf{nat} \wedge \forall x\,\varepsilon\,\mathsf{nat}(\mathsf{s_N}x\,\varepsilon\,\mathsf{nat})$;
(5) $\forall x\,\varepsilon\,\mathsf{nat}(\mathsf{s_N}x \neq 0 \wedge \mathsf{p_N}(\mathsf{s_N}x) = x)$;
(6) $\forall x\,\varepsilon\,\mathsf{nat}(x \neq 0 \to \mathsf{p_N}x\,\varepsilon\,\mathsf{nat} \wedge \mathsf{s_N}(\mathsf{p_N}x) = x)$;
(7) $a\,\varepsilon\,\mathsf{nat} \wedge b\,\varepsilon\,\mathsf{nat} \to (a = b \to \mathsf{d_N}xyab = x) \wedge (a \neq b \to \mathsf{d_N}xyab = y)$.

II. Induction on nat.
$F[0] \wedge \forall x(F[x] \to F[\mathsf{s_N}x]) \to \forall x\,\varepsilon\,\mathsf{nat}F[x]$
for each formula F.
We will also consider restricted form of induction, where $F[x]$ must be of the form $x\,\varepsilon\,\gamma$.

The following lemmas 1.1 and 1.2 are provable using only applicative axioms I; Lemma 1.3 in addition calls for restricted induction on natural numbers (see, for example, [Fef79], [Be85], or a review [JKS99]).

Lemma 1.1 λ-abstraction
For every term $t[x]$ there exists a term $\lambda x.t[x]$ such that $\lambda x.t[x]\downarrow$ and for every term s

$$s\downarrow \to (\lambda x.t[x])s \simeq t[s]).$$

Abbreviation. We will use ID for $\langle \lambda x.x, \lambda y.y\rangle$.

Lemma 1.2 Recursion Theorem
There exists a closed term rec *such that*

$$\mathsf{rec}f\downarrow \wedge\ \mathsf{rec}fx \simeq f(\mathsf{rec}f)x.$$

Lemma 1.3 Primitive recursion on natural numbers
There exists a closed term prim *such that*

$$f : \mathsf{nat} \mapsto \mathsf{nat} \wedge g : \mathsf{nat}^3 \mapsto \mathsf{nat} \wedge x\,\varepsilon\,\mathsf{nat} \wedge y\,\varepsilon\,\mathsf{nat} \ \to$$
$$\mathsf{prim}fg : \mathsf{nat}^2 \mapsto \mathsf{nat} \wedge \mathsf{prim}fgx0 = fx \wedge \mathsf{prim}fgx(\mathsf{s_N}y) = gxy(\mathsf{prim}fgxy).$$

III. Explicit representation. This axiom states that each name is an operation:
$\exists x(x = \alpha)$.

IV. Elementary comprehension (ECA). These axiomatise *name generators*:
(1) $\mathcal{N}[\mathsf{nat}]$;
(2) $\mathcal{N}[\mathsf{id}] \wedge \forall x(x\,\varepsilon\,\mathsf{id} \leftrightarrow x = \langle \mathsf{p_0}x, \mathsf{p_1}x\rangle \wedge \mathsf{p_0}x = \mathsf{p_1}x)$;

(3) $\mathcal{N}[\mathsf{inv}(f,\alpha)] \wedge \forall x(x \,\varepsilon\, \mathsf{inv}(f,\alpha) \leftrightarrow fx \,\varepsilon\, \alpha)$;
(4) $\mathcal{N}[\mathsf{and}(\alpha,\beta)] \wedge \forall x(x \,\varepsilon\, \mathsf{and}(\alpha,\beta) \leftrightarrow x \,\varepsilon\, \alpha \wedge x \,\varepsilon\, \beta)$;
(5) $\mathcal{N}[\mathsf{or}(\alpha,\beta)] \wedge \forall x(x \,\varepsilon\, \mathsf{or}(\alpha,\beta) \leftrightarrow x \,\varepsilon\, \alpha \vee x \,\varepsilon\, \beta)$;
(6) $\mathcal{N}[\mathsf{imp}(\alpha,\beta)] \wedge \forall x(x \,\varepsilon\, \mathsf{imp}(\alpha,\beta) \leftrightarrow x \,\varepsilon\, \alpha \to x \,\varepsilon\, \beta)$;
(7) $\mathcal{N}[\mathsf{all}\alpha] \wedge \forall x(x \,\varepsilon\, \mathsf{all}\alpha \leftrightarrow \forall y(\langle x,y\rangle \,\varepsilon\, \alpha))$;
(8) $\mathcal{N}[\mathsf{ex}\alpha] \wedge \forall x(x \,\varepsilon\, \mathsf{ex}\alpha \leftrightarrow \exists y(\langle x,y\rangle \,\varepsilon\, \alpha))$.

Definition 1.1 Elementary formula
A formula is elementary *iff it's constructed from $s = t$ and $t \,\varepsilon\, \alpha$ by means of $\wedge, \vee, \to, \forall x, \exists x$ only. (No occurrences of $t \,\varepsilon\, s$ with s not a name variable and name quantifiers are allowed.)*

The following lemma is an intuitionistic analogue of reducing Elementary Comprehension as stated in [Fef75] to name generators nat, id, co, int, dom and inv, which holds in classical setting (see [FJ96]); its proof requires only axioms I, III and IV. For alternative intuitionistic reductions of Elementary Comprehension to a finite number of its instances see [GR94, Sect.1] and [Tat98, Sect.3].

Lemma 1.4 ECA
If a formula $F := F[x;\bar{a};\bar{\alpha}]$ is elementary then there exists a term t_F^x such that $\mathrm{FV}(\mathsf{t}_F^x) = \mathrm{FV}(F) \setminus \{x\}$ and

$$\mathcal{N}[\mathsf{t}_F^x] \wedge \forall x(x \,\varepsilon\, \mathsf{t}_F^x \leftrightarrow F).$$

Proof. The term t_F^x is built by recursion on F:

$$\mathsf{t}_F^x := \begin{cases} \mathsf{inv}(\lambda x.\langle s[x], t[x]\rangle, \mathsf{id}) & \text{if } F \text{ is } s[x] = t[x]; \\ \mathsf{inv}(\lambda x.s[x], \alpha) & \text{if } F \text{ is } s[x] \,\varepsilon\, \alpha; \\ \mathsf{inv}(\lambda x.\langle s[x], s[x]\rangle, \mathsf{id}) & \text{if } F \text{ is } s[x]\downarrow; \\ \mathsf{and}(\mathsf{t}^x_{F_0[x]}, \mathsf{t}^x_{F_1[x]}) & \text{if } F \text{ is } F_0[x] \wedge F_1[x]; \\ \mathsf{or}(\mathsf{t}^x_{F_0[x]}, \mathsf{t}^x_{F_1[x]}) & \text{if } F \text{ is } F_0[x] \vee F_1[x]; \\ \mathsf{imp}(\mathsf{t}^x_{F_0[x]}, \mathsf{t}^x_{F_1[x]}) & \text{if } F \text{ is } F_0[x] \to F_1[x]; \\ \mathsf{allt}^z_{G[\mathsf{p}_0 z, \mathsf{p}_1 z]} & \text{if } F \text{ is } \forall y G[x,y]; \\ \mathsf{ext}^z_{G[\mathsf{p}_0 z, \mathsf{p}_1 z]} & \text{if } F \text{ is } \exists y G[x,y]. \end{cases}$$

Now the property of t_F^x is proved by induction on F.

□

A formula F is *elementary in $\bar{t}$* iff F is a substitution instance of a list $\bar{t}$ of names for name variables into an elementary formula. Most often, when name parameters are clear, we will call an *elementary in $\bar{t}$* formula plainly *elementary*.

V. Join (J). This axiom states that if f is an operation from a type named by α, each value of which is a name, then $\mathsf{j}(\alpha, f)$ names a disjoint union of all fx for $x \,\varepsilon\, \alpha$:
$\forall x \,\varepsilon\, \alpha \mathcal{N}[fx] \to \big(\mathcal{N}[\mathsf{j}(\alpha,f)] \;\wedge\; \forall z(z \,\varepsilon\, \mathsf{j}(\alpha,f) \leftrightarrow z = \langle \mathsf{p}_0 z, \mathsf{p}_1 z\rangle \wedge \mathsf{p}_0 z \,\varepsilon\, \alpha \wedge y \,\varepsilon\, fx)\big)$.

VI. Inductive Generation (IG). The first part of this axiom states that $\mathsf{i}(\alpha,\beta)$ names a wellfounded part of a type named by α along an ordering named by β; the second part allows induction over that type for an arbitrary formula:
$\mathcal{N}[\mathsf{i}(\alpha,\beta)] \;\wedge\; \forall x \,\varepsilon\, \alpha \left(\forall y(\langle y,x\rangle \,\varepsilon\, \beta \to y \,\varepsilon\, \mathsf{i}(\alpha,\beta)) \to x \,\varepsilon\, \mathsf{i}(\alpha,\beta)\right)$
$\wedge\; \left(\forall x \,\varepsilon\, \alpha \left(\forall y(\langle y,x\rangle \,\varepsilon\, \beta \to F[y]) \to F[x]\right) \to \forall x \,\varepsilon\, \mathsf{i}(\alpha,\beta) F[x]\right)$,
where F is an arbitrary formula.
By **IG**$\restriction$ (**IG** restricted) we denote the schema **IG** with F only of the form $x \,\varepsilon\, \gamma$.

The theory **App** is the one containing only applicative axioms I; **EON** has axioms I–II. The theory **EONN** has axioms of the groups I–III. **EET** is **EONN** + **ECA**, **EETJ** is **EET** + **J** and $\mathbf{T_0}$ is **EETJ** + **IG**. These theories are formulated in the language $\mathcal{L}_{\mathbf{T_0}}$. The remaining theories $\mathbf{T_u}$, $\mathbf{T_m}$ and $\mathbf{T_M}$ are formulated in the languages $\mathcal{L}_{\mathbf{T_u}}$, $\mathcal{L}_{\mathbf{T_m}}$ and $\mathcal{L}_{\mathbf{T_M}}$, respectively. By $\mathbf{T}\restriction$ we mean a version of a theory $\mathbf{T} \in \mathcal{L}_{\mathbf{T_M}}$ where both induction on natural numbers II and inductive generation VI, if applicable, are restricted. $(\mathbf{EET} + \mathbf{IG})\restriction$ is $\mathbf{EET}\restriction + \mathbf{IG}\restriction$.

To state theories $\mathbf{T_u}$, $\mathbf{T_m}$ and $\mathbf{T_M}$ of Explicit Mathematics, we need a notion of a *universe*.

Definition 1.2 t names a universe, $\mathcal{U}[t]$
We denote by $\mathcal{U}[t]$ *the following formula:*

$$\mathcal{N}[t] \wedge \forall x \,\varepsilon\, t \mathcal{N}[x] \wedge$$
$$\mathsf{nat}, \mathsf{id} \,\varepsilon\, t \wedge (\alpha, \beta \,\varepsilon\, t \wedge f : \alpha \mapsto t \;\rightarrow\; \mathsf{inv} g\alpha, \mathsf{and}\alpha\beta, \mathsf{or}\alpha\beta, \mathsf{imp}\alpha\beta, \mathsf{all}\alpha, \mathsf{ex}\alpha, \mathsf{j}\alpha f \,\varepsilon\, t).$$

According to this definition, *universes are types, closed under name generators and join.*

VII. Universe operation (u)**.** This axiom says that given a name α, $\mathsf{u}\alpha$ names a universe containing α.
The theory $\mathbf{T_u}$ has axioms I–VI plus the following axiom:
$\mathcal{U}(\mathsf{u}\alpha) \wedge \alpha \,\varepsilon\, \mathsf{u}\alpha$.
This theory has been proved in [JStu] (called $\mathsf{T_0} + (\mathsf{Lim})$ there) to have the same strength as $\mathbf{T_0}$.

VIII. Mahlo operation (m). This axiom states that given an operation f from names to names, $\mathsf{m}f$ names a universe closed under f.
The theory $\mathbf{T_m}$ has axioms I–VII plus the following axiom:
$\forall \beta \mathcal{N}[f\beta] \rightarrow \mathcal{U}(\mathsf{m}f) \wedge f : \mathsf{m}f \mapsto \mathsf{m}f$.
Remark. We divided the Mahlo axiom as introduced in [JStu] into the limit axiom VII and properly Mahlo axiom VIII, which has to do with functions only. This is a minor modification, not playing any essential role.

IX. Mahlo universe (M)**.** This axiom asserts that a universe M is closed under Mahlo operation m.
The theory $\mathbf{T_M}$ has axioms I–VII plus the following axiom:
$\mathcal{U}[\mathsf{M}] \wedge \mathsf{u} : \mathsf{M} \mapsto \mathsf{M} \wedge \mathsf{i} : \mathsf{M}^2 \mapsto \mathsf{M} \wedge \forall f : \mathsf{M} \mapsto \mathsf{M}\ (\mathcal{U}[\mathsf{m}f] \wedge f : \mathsf{m}f \mapsto \mathsf{m}f \wedge \mathsf{m}f \,\varepsilon\, \mathsf{M})$.

2 Sets in Explicit Mathematics. Realization

The language $\mathcal{L}_\in$ of Set Theory is first-order with set variables $A, B, C, I, R, S, T, U, V, W, X, Y, Z$ and two predicate constants $=$ and $\in$. For each set variable $A \in \mathcal{L}_\in$ we assume a name variable $\alpha_A \in \mathcal{L}_{\mathbf{T_0}}$.
As usual, $\forall X \in A F[X]$ and $\exists X \in A F[X]$ stand for $\forall X (X \in A \rightarrow F[X])$ and $\exists X (X \in A \wedge F[X])$, respectively. By F^A we denote the result of replacing each quantifier QX in F by $QX \in A$. *Bounded* formulas of $\mathcal{L}_\in$ are those built from atoms by means of $\wedge$, $\vee$, $\rightarrow$, $\forall X \in A$ and $\exists X \in A$. We will use the same syntactical conventions as in Section 1.
Sets are interpreted as (names of) wellfounded trees. To begin, we need to set the stage.
We can define a name seq of the type of sequences so that

$$\mathsf{seq} \doteq \{\langle x, y\rangle \mid (x = 0 \wedge y = 0) \vee (x = 1 \wedge y = \langle \mathsf{p_0} y, \mathsf{p_1} y\rangle \wedge \mathsf{p_0} y \,\varepsilon\, \mathsf{seq})\}. \tag{2.1}$$

To do this, by **ECA** one defines a name $\mathsf{seq_0}$ s.t.

$$\mathsf{seq_0} \doteq \{\langle 0, 0\rangle\}, \tag{2.2}$$

and an operation $\mathsf{seq_S}$ s.t.

$$\mathsf{seq_S}\alpha \doteq \{\langle 1, \langle y, z\rangle\rangle \mid y \,\varepsilon\, \alpha\}. \tag{2.3}$$

Then by primitive recursion one defines an operation sq s.t.

$$\begin{cases} \mathsf{sq}0 = \mathsf{seq_0}, \\ \mathsf{sq}(n') = \mathsf{seq_S}(\mathsf{sq}n). \end{cases} \tag{2.4}$$

Finally by **ECA** one sets

$$\mathsf{seq} :\doteq \{x \mid \exists n \,\varepsilon\, \mathsf{nat}(x \,\varepsilon\, \mathsf{sq}n)\}. \tag{2.5}$$

We abbreviate

$$\mathsf{nil} := \langle 0, 0\rangle. \tag{2.6}$$

One defines a *length* operation ln by recursion theorem to satisfy the following equations:

$$\begin{cases} \mathsf{ln}\mathsf{nil} \simeq 0, \\ \mathsf{ln}\langle 1, \langle a, b\rangle\rangle \simeq \mathsf{s_N}(\mathsf{ln}a). \end{cases} \tag{2.7}$$

The following proposition is immediate from the Definitions:

Proposition 2.1 *If* $a \,\varepsilon\, \mathsf{seq}$ *then* $\mathsf{ln}a \,\varepsilon\, \mathsf{nat}$*. Given* $a \,\varepsilon\, \mathsf{seq}$*, we have* $a = \mathsf{nil} \leftrightarrow \mathsf{ln}a = 0$.

Concatenation operation conc is defined by the following equations:

$$\begin{cases} \mathsf{conc}(a, \mathsf{nil}) \simeq a, \\ \mathsf{conc}(a, \langle 1, \langle c, d \rangle\rangle) \simeq \langle 1, \langle \mathsf{conc}(a, c), d\rangle\rangle. \end{cases} \tag{2.8}$$

Again we have the following proposition:

Proposition 2.2 *If* $a \,\varepsilon\, \mathsf{seq}$*,* $b \,\varepsilon\, \mathsf{seq}$ *and* $c \,\varepsilon\, \mathsf{seq}$ *then the following holds:*
a) $\mathsf{conc}(a, b) \,\varepsilon\, \mathsf{seq}$ *and* $\mathsf{ln}(\mathsf{conc}(a, b)) = \mathsf{ln}a + \mathsf{ln}b$*;*
b) $\mathsf{conc}(a, \mathsf{nil}) = \mathsf{conc}(\mathsf{nil}, a) = a$*;*
c) $\mathsf{conc}(a, \mathsf{conc}(b, c)) = \mathsf{conc}(\mathsf{conc}(a, b), c)$*.*

$a * b$ will be used for $\mathsf{conc}(a, b)$.

We need to define head, tail and body of a sequence:

$$\begin{cases} \mathsf{head}\langle 1, \langle \mathsf{nil}, d\rangle\rangle \simeq \langle 1, \langle \mathsf{nil}, d\rangle\rangle, \\ \mathsf{head}\langle 1, \langle c, d\rangle\rangle \simeq \mathsf{head}c \qquad \text{if } \mathsf{ln}c \neq 0; \end{cases} \tag{2.9}$$

$$\mathsf{tail}\langle 1, \langle c, d\rangle\rangle \simeq \langle 1, \langle \mathsf{nil}, d\rangle\rangle; \tag{2.10}$$

$$\begin{cases} \mathsf{body}\mathsf{nil} \simeq \mathsf{nil}, \\ \mathsf{body}\langle 1, \langle \mathsf{nil}, d\rangle\rangle \simeq \mathsf{nil}, \\ \mathsf{body}\langle 1, \langle c, d\rangle\rangle \simeq \mathsf{conc}(\mathsf{body}c, \mathsf{tail}c) \quad \text{if } \mathsf{ln}c \neq 0. \end{cases} \tag{2.11}$$

From the definitions 2.9–2.11 we have the following proposition:

Proposition 2.3 *If* $a \,\varepsilon\, \mathsf{seq}$ *then the following holds:*
a) $\mathsf{body}a \,\varepsilon\, \mathsf{seq} \wedge (\mathsf{ln}a \neq 0 \to \mathsf{head}a, \mathsf{tail}a \,\varepsilon\, \mathsf{seq})$*;*
b) $\mathsf{ln}a = 1 \to a = \mathsf{head}a = \mathsf{tail}a$*;*
c) $\mathsf{ln}a \neq 0, 1 \to a = \mathsf{conc}(\mathsf{head}a, \mathsf{conc}(\mathsf{body}a, \mathsf{tail}a))$*.*

We set

$$\mathsf{hbody}a := \mathsf{conc}(\mathsf{head}a, \mathsf{body}a); \qquad \mathsf{bodyt}a := \mathsf{conc}(\mathsf{body}a, \mathsf{tail}a). \tag{2.12}$$

We define operations fcut and cutb, which cut members from the beginning and the end of a sequence, by the following equations:

$$\begin{cases} \mathsf{fcut}(c, 0) \simeq c, \\ \mathsf{fcut}(c, \mathsf{s_N}n) \simeq \mathsf{bodyt}(\mathsf{fcut}(c, n)); \end{cases} \qquad \begin{cases} \mathsf{cutb}(c, 0) \simeq c, \\ \mathsf{cutb}(c, \mathsf{s_N}n) \simeq \mathsf{hbody}(\mathsf{cutb}(c, n)). \end{cases} \tag{2.13}$$

Again we have the following proposition:

Proposition 2.4 *If* $a \,\varepsilon\, \mathsf{seq}$ *and* $0 \leq n \leq \mathsf{ln}a$ *then* $\mathsf{fcut}(a, n), \mathsf{cutb}(a, n) \,\varepsilon\, \mathsf{seq}$ *and* $\mathsf{ln}(\mathsf{fcut}(a, n)) = \mathsf{ln}(\mathsf{cutb}(a, n)) = \mathsf{ln}a - n$*.*

n-th element of a sequence is defined as follows:

$$\mathsf{el}(c, n) := \mathsf{p_1}(\mathsf{p_1}(\mathsf{head}(\mathsf{fcut}(c, \mathsf{p_N}n)))) \tag{2.14}$$

We have:

Proposition 2.5 *If* $a \,\varepsilon\, \mathsf{seq}$ *and* $0 < n \leq \mathsf{ln}a$ *then* $\mathsf{el}(a, n)\downarrow$*.*

We define operations sg *(singleton)* and ct *(content)* by the following equations:

$$\mathsf{sg} := \lambda x.\langle 1, \langle \mathsf{nil}, x\rangle\rangle \tag{2.15}$$

and

$$\mathsf{ct} := \lambda x.\mathsf{el}(x, 1). \tag{2.16}$$

Finally, we need to define a "smash" operation sm and its inverses $\mathsf{sm_0}$ and $\mathsf{sm_1}$: $\mathsf{sm}(x, y)$ tags y to every element of a sequence x.

$$\mathsf{sm}(\langle 1, \langle a, b\rangle\rangle, y) \simeq \langle 1, \langle \mathsf{sm}(a, y), \langle b, y\rangle\rangle\rangle, \tag{2.17}$$

$$\mathsf{sm_0}\langle 1, \langle a, b\rangle\rangle \simeq \langle 1, \langle \mathsf{sm_0}a, \mathsf{p_0}b\rangle\rangle, \qquad \mathsf{sm_1}\langle 1, \langle a, b\rangle\rangle \simeq \mathsf{p_1}b. \tag{2.18}$$

Proposition 2.6 *If* $x \varepsilon \mathsf{seq}$ *and* $\mathsf{ln}x \neq 0$ *then* $\mathsf{sm}(x,y), \mathsf{sm}_0 x \varepsilon \mathsf{seq}$, $\mathsf{ln}(\mathsf{sm}(x,y)) = \mathsf{ln}(\mathsf{sm}_0 x) = \mathsf{ln}x$ *and* $\mathsf{sm}_1 x\downarrow$.

Definition 2.1 $\sqsupset$
By **ECA** *a name* $\sqsupset$ *is defined so that*

$$\sqsupset \doteq \{\langle x,y\rangle \mid x \varepsilon \mathsf{seq} \wedge y \varepsilon \mathsf{seq} \wedge \exists z \varepsilon \mathsf{seq}(\mathsf{ln}z \neq 0 \wedge y{*}z = x)\}. \tag{2.19}$$

We will use $x \sqsupset y$, $x \sqsupseteq y$ in place of $\langle x,y\rangle \varepsilon \sqsupset$ and $(x \varepsilon \mathsf{seq} \wedge x = y) \vee x \sqsupset y$, resp.
The following two propositions are immediate from 2.19.

Proposition 2.7 *If* $x \sqsupset y$ *then* $\mathsf{ln}x > \mathsf{ln}y$.

Proposition 2.8 *If* $x \sqsupseteq y$ *then* $x = y{*}\mathsf{fcut}(x, \mathsf{ln}y)$.

Definition 2.2 ig
ig *is defined as* $\lambda\alpha.\mathsf{i}(\alpha, \sqsupset)$.

A *set* is a *wellfounded tree*, i.e. non-empty type of sequences, downwards closed and contained in its well-founded part with respect to $\sqsupset$-relation:

Definition 2.3 t names a set, $\mathrm{Set}[t]$
$\mathrm{Set}[t]$ *is defined as*

$$\mathcal{N}[t] \wedge t \dot{\subseteq} \mathsf{seq} \wedge \mathsf{nil} \varepsilon t \wedge \forall x \varepsilon t \forall y(x \sqsupset y \to y \varepsilon t) \wedge t \dot{\subseteq} \mathsf{ig}t. \tag{2.20}$$

Note 1. By $\mathbf{IG}\restriction$ we have $\mathrm{Set}[t] \to t \doteq \mathsf{ig}t$.
Note 2. We needed sets to be wellfounded trees consisting of *any* objects, not natural numbers or any other special kinds. The reason is interpreting Mahlo axioms. For weaker theories, some special kinds of trees could suffice.

Definition 2.4 Subtree operation, str
By **ECA** *we define an operation* str *in such a way that*

$$\mathcal{N}[\mathsf{str}(\alpha,z)] \wedge (x \varepsilon \mathsf{str}(\alpha,z) \leftrightarrow x \varepsilon \mathsf{seq} \wedge z{*}x \varepsilon \alpha). \tag{2.21}$$

Lemma 2.1 *In* $(\mathbf{EET}+\mathbf{IG})\restriction$ *we have*

$$\mathrm{Set}[\alpha] \wedge z \varepsilon \alpha \to \mathrm{Set}[\mathsf{str}(\alpha,z)]. \tag{2.22}$$

Proof. Given $\mathrm{Set}[\alpha]$, by induction on $\mathsf{ig}\alpha \doteq \alpha$, we prove

$$\forall z \varepsilon \alpha(\mathsf{str}(\alpha,z) \dot{\subseteq} \mathsf{igstr}(\alpha,z)).$$

Non-emptiness and downwards closeness of $\mathsf{str}(\alpha,z)$ follow vacuously from 2.21.

□

Definition 2.5 Bisimulation
A formula $BS[\mathfrak{r},\alpha,\beta]$ *is defined as*

$$\begin{array}{l} \forall x \varepsilon \alpha \left(\mathsf{p}_0\mathfrak{r}x \varepsilon \beta \wedge \mathsf{ln}(\mathsf{p}_0\mathfrak{r}x) = \mathsf{ln}(x) \wedge \right. \\ \qquad \left. \forall x' \varepsilon \alpha(x' \sqsupseteq x \to \mathsf{p}_0\mathfrak{r}x' \sqsupseteq \mathsf{p}_0\mathfrak{r}x) \wedge \forall y \varepsilon \beta(y \sqsupseteq \mathsf{p}_0\mathfrak{r}x \to \mathsf{p}_1\mathfrak{r}y \sqsupseteq x)\right) \bigwedge \\ \forall y \varepsilon \beta \left(\mathsf{p}_1\mathfrak{r}y \varepsilon \alpha \wedge \mathsf{ln}(\mathsf{p}_1\mathfrak{r}y) = \mathsf{ln}(y) \wedge \right. \\ \qquad \left. \forall y' \varepsilon \beta(y' \sqsupseteq y \to \mathsf{p}_1\mathfrak{r}y' \sqsupseteq \mathsf{p}_1\mathfrak{r}y) \wedge \forall x \varepsilon \alpha(x \sqsupseteq \mathsf{p}_1\mathfrak{r}y \to \mathsf{p}_0\mathfrak{r}x \sqsupseteq y)\right). \end{array} \tag{2.23}$$

An $\mathfrak{r}$ *such that* $BS[\mathfrak{r},\alpha,\beta]$ *is called* bisimulator *for* α *and* β *and in this case* α *and* β *are called* bisimulable by $\mathfrak{r}$.

Lemma 2.2 Bisimulation is an equivalence relation
a) $BS[\mathsf{ID},\alpha,\alpha]$;
b) $BS[\mathfrak{r},\alpha,\beta] \to BS[\langle \mathsf{p}_1\mathfrak{r}, \mathsf{p}_0\mathfrak{r}\rangle, \beta, \alpha]$;
c) $BS[\mathfrak{r},\alpha,\beta] \wedge BS[\mathfrak{s},\beta,\gamma] \to BS[\langle \lambda x.\mathsf{p}_0\mathfrak{s}(\mathsf{p}_0\mathfrak{r}x), \lambda z.\mathsf{p}_1\mathfrak{r}(\mathsf{p}_1\mathfrak{s}z)\rangle, \alpha, \gamma]$.

Proof. Immediate from Definition 2.5. $\square$

Lemma 2.3 *There exists an operation* sbs *such that* $\mathsf{sbs}\downarrow$ *and if* $\mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta] \wedge BS[\mathfrak{r}, \alpha, \beta] \wedge z \,\varepsilon\, \alpha$ *then* $\mathsf{sbs}\mathfrak{r}z\downarrow$ *and* $BS[\mathsf{sbs}\mathfrak{r}z, \mathsf{str}(\alpha, z), \mathsf{str}(\beta, \mathsf{p}_0\mathfrak{r}z)]$.

Proof. 1). Assume $x \,\varepsilon\, \mathsf{str}(\alpha, z)$. By 2.21 this means

$$x'[x, z] := z{*}x \,\varepsilon\, \alpha.$$

We have $x' \sqsupseteq z$, so by 2.23 $\mathsf{p}_0\mathfrak{r}x' \,\varepsilon\, \beta \wedge \mathsf{p}_0\mathfrak{r}x' \sqsupseteq \mathsf{p}_0\mathfrak{r}z$. By Proposition 2.8

$$\mathsf{p}_0\mathfrak{r}x' = \mathsf{p}_0\mathfrak{r}z{*}\mathsf{fcut}(\mathsf{p}_0\mathfrak{r}x', \mathsf{ln}(\mathsf{p}_0\mathfrak{r}z)) \,\varepsilon\, \beta,$$

i.e.

$$\mathsf{fcut}(\mathsf{p}_0\mathfrak{r}x', \mathsf{ln}(\mathsf{p}_0\mathfrak{r}z)) \,\varepsilon\, \mathsf{str}(\beta, \mathsf{p}_0\mathfrak{r}z). \tag{2.24}$$

2). Symmetrically, assume $y \,\varepsilon\, \mathsf{str}(\beta, \mathsf{p}_0\mathfrak{r}z)$. By 2.21 this means

$$y'[\mathfrak{r}, y, z] := \mathsf{p}_0\mathfrak{r}z{*}y \,\varepsilon\, \beta.$$

We have $y' \sqsupseteq \mathsf{p}_0\mathfrak{r}z$, so by 2.23 $\mathsf{p}_1\mathfrak{r}y' \,\varepsilon\, \alpha \wedge \mathsf{p}_1\mathfrak{r}y' \sqsupseteq z$. By Proposition 2.8

$$\mathsf{p}_1\mathfrak{r}y' = z{*}\mathsf{fcut}(\mathsf{p}_1\mathfrak{r}y', \mathsf{ln}z) \,\varepsilon\, \alpha,$$

i.e.

$$\mathsf{fcut}(\mathsf{p}_1\mathfrak{r}y', \mathsf{ln}z) \,\varepsilon\, \mathsf{str}(\alpha, z). \tag{2.25}$$

From 2.24 and 2.25 for

$$\mathsf{sbs}'[\mathfrak{r}, z] := \langle \lambda x.\mathsf{fcut}(\mathsf{p}_0\mathfrak{r}x', \mathsf{ln}(\mathsf{p}_0\mathfrak{r}z)), \lambda y.\mathsf{fcut}(\mathsf{p}_1\mathfrak{r}y', \mathsf{ln}z)\rangle$$

we have

$$\forall x \,\varepsilon\, \mathsf{str}(\alpha, z)(\mathsf{p}_0\mathsf{sbs}'x \,\varepsilon\, \mathsf{str}(\beta, \mathsf{p}_0\mathfrak{r}z)) \wedge \forall y \,\varepsilon\, \mathsf{str}(\beta, \mathsf{p}_0\mathfrak{r}z)(\mathsf{p}_1\mathsf{sbs}'y \,\varepsilon\, \mathsf{str}(\alpha, z)).$$

Other conditions 2.23 for $BS[\mathsf{sbs}', \mathsf{str}(\alpha, z), \mathsf{str}(\beta, \mathsf{p}_0\mathfrak{r}z)]$ follow from corresponding conditions for $BS[\mathfrak{r}, \alpha, \beta]$. Finally we set $\mathsf{sbs} := \lambda\mathfrak{r}\lambda z.\mathsf{sbs}'[\mathfrak{r}, z]$.

$\square$

Definition 2.6 $\mathfrak{r}$ realizes F, $\mathfrak{r}\ \underline{\mathrm{rn}}\ F$

For each formula $F \in \mathcal{L}_\in$ *we define a formula* $(\mathfrak{r}\ \underline{\mathrm{rn}}\ F) \in \mathcal{L}_{\mathbf{T_0}}$ *with a new free individual variable* $\mathfrak{r}$. *The definition is given by the table below:*

$F \in \mathcal{L}_\in$	$(\mathfrak{r}\ \underline{\mathrm{rn}}\ F) \in \mathcal{L}_{\mathbf{T_0}}$
$A = B$	$BS[\mathfrak{r}, \alpha_A, \alpha_B]$
$A \in B$	$\mathsf{p}_0\mathfrak{r} \,\varepsilon\, \alpha_B \wedge \mathsf{ln}(\mathsf{p}_0\mathfrak{r}) = 1 \wedge BS[\mathsf{p}_1\mathfrak{r}, \alpha_A, \mathsf{str}(\alpha_B, \mathsf{p}_0\mathfrak{r})]$
$F_0 \wedge F_1$	$\mathsf{p}_0\mathfrak{r}\ \underline{\mathrm{rn}}\ F_0 \wedge \mathsf{p}_1\mathfrak{r}\ \underline{\mathrm{rn}}\ F_1$
$F_0 \vee F_1$	$\mathsf{p}_0\mathfrak{r} \,\varepsilon\, \mathsf{nat} \ \wedge \ \begin{array}{l}(\mathsf{p}_0\mathfrak{r} = 0 \to \mathsf{p}_1\mathfrak{r}\ \underline{\mathrm{rn}}\ F_0) \ \wedge \\ (\mathsf{p}_0\mathfrak{r} \neq 0 \to \mathsf{p}_1\mathfrak{r}\ \underline{\mathrm{rn}}\ F_1)\end{array}$
$F_0 \to F_1$	$\forall\mathfrak{x}(\mathfrak{x}\ \underline{\mathrm{rn}}\ F_0 \to \mathfrak{r}\mathfrak{x}\downarrow \wedge \ \mathfrak{r}\mathfrak{x}\ \underline{\mathrm{rn}}\ F_1)$
$\forall X G[X]$	$\forall\alpha(\mathrm{Set}[\alpha] \to \mathfrak{r}\alpha\downarrow \wedge \ \mathfrak{r}\alpha\ \underline{\mathrm{rn}}\ G[\alpha])$
$\exists X G[X]$	$\mathrm{Set}[\mathsf{p}_0\mathfrak{r}] \wedge \mathsf{p}_1\mathfrak{r}\ \underline{\mathrm{rn}}\ G[\mathsf{p}_0\mathfrak{r}]$

Remark. According to our notation for substitution, p. 3, in the previous definition $\mathsf{p}_1\mathfrak{r}\ \underline{\mathrm{rn}}\ G[\mathsf{p}_0\mathfrak{r}]$, for example, stands for $(\mathfrak{r}\ \underline{\mathrm{rn}}\ G[X])^{\mathsf{p}_1\mathfrak{r}, \mathsf{p}_0\mathfrak{r}}_{\mathfrak{r}, \alpha_X}$.

Definition 2.7 $\mathcal{R}$-interpretation
For each $F \in \mathcal{L}_\in$ *we define* $\mathcal{R}(F) := \exists\mathfrak{x}(\mathfrak{x}\ \underline{\mathrm{rn}}\ F)$.

Definition 2.8 Realization, realizable

1. *A term* $\mathsf{t} \in \mathcal{L}_{\mathbf{T_m}}$ *is called* realization *of a formula* $F \in \mathcal{L}_\in$ *in a theory* $\mathbf{T}$, *iff*

$$\mathrm{FV}(\mathsf{t}) \subseteq \{\alpha_A \mid A \in \mathrm{FV}(F)\}$$

and

$$\mathbf{T} \vdash \bigwedge_{A \in \mathrm{FV}(F)} \mathrm{Set}[\alpha_A] \to \mathsf{t}\ \underline{\mathrm{rn}}\ F.$$

2. *If there exists such a term* t *then* F *is called* realizable in $\mathbf{T}$. *We call a theory* $\mathrm{T_S}$ realizable in $\mathbf{T}$ *iff every theorem of* $\mathrm{T_S}$ *is realizable in* $\mathbf{T}$. $\mathrm{T_S}$ *is* realizable *iff it's realizable in* $(\mathbf{EET}+\mathbf{IG})\!\upharpoonright$.

Note 1. If F is closed and realizable in $\mathbf{T}$ then $\mathbf{T} \vdash \mathcal{R}(F)$.
Note 2. $\alpha \doteq \beta \leftrightarrow \mathsf{ID}\ \underline{\mathrm{rn}}\ (\alpha = \beta)$.

Theorem 1 *Each theorem of intuitionistic first-order predicate calculus with equality is realizable in* $(\mathbf{EET}+\mathbf{IG})\!\upharpoonright$.

The **proof** is standard except for the case of equality axioms. We need to build realizations for the following axioms:
(Eq1) $A = A$;
(Eq2) $A = B \to B = A$;
(Eq3) $A = B \wedge B = C \to A = C$;
(Eq4) $A = B \wedge C \in A \to C \in B$;
(Eq5) $A = B \wedge A \in C \to B \in C$.
Lemma 2.2 provides realizations for (Eq1)–(Eq3). For (Eq4) and (Eq5), assume commonly $\mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta] \wedge \mathrm{Set}[\gamma] \wedge BS[\mathfrak{r}, \alpha, \beta]$.
For (Eq4), assume $\mathsf{p_0}\mathfrak{s}\ \varepsilon\ \alpha \wedge \mathsf{ln}(\mathsf{p_0}\mathfrak{s}) = 1 \wedge BS[\mathsf{p_1}\mathfrak{s}, \gamma, \mathsf{str}(\alpha, \mathsf{p_0}\mathfrak{s})]$. Then $\mathsf{p_0}\mathfrak{r}(\mathsf{p_0}\mathfrak{s})\ \varepsilon\ \beta \wedge \mathsf{ln}(\mathsf{p_0}\mathfrak{r}(\mathsf{p_0}\mathfrak{s})) = 1$ and by Lemma 2.3

$$BS[\mathsf{sbs}(\mathfrak{r}, \mathsf{p_0}\mathfrak{s}), \mathsf{str}(\alpha, \mathsf{p_0}\mathfrak{s}), \mathsf{str}(\beta, \mathsf{p_0}\mathfrak{r}(\mathsf{p_0}\mathfrak{s}))].$$

By transitivity (Lemma 2.2c)

$$BS[\langle \lambda z.\mathsf{p_0}\mathsf{sbs}(\mathfrak{r}, \mathsf{p_0}\mathfrak{s})(\mathsf{p_0}(\mathsf{p_1}\mathfrak{s})z), \lambda y.\mathsf{p_1}(\mathsf{p_1}\mathfrak{s})(\mathsf{p_1}\mathsf{sbs}(\mathfrak{r}, \mathsf{p_0}\mathfrak{s})y)\rangle, \gamma, \mathsf{str}(\beta, \mathsf{p_0}\mathfrak{r}(\mathsf{p_0}\mathfrak{s}))],$$

which gives a realization of (Eq4).
For (Eq5), assume $\mathsf{p_0}\mathfrak{s}\,\varepsilon\,\gamma \wedge \mathsf{ln}(\mathsf{p_0}\mathfrak{s}) = 1 \wedge BS[\mathsf{p_1}\mathfrak{s}, \alpha, \mathsf{str}(\gamma, \mathsf{p_0}\mathfrak{s})]$. By symmetry and transitivity (Lemma 2.2b,c) we have

$$BS[\langle \lambda y.\mathsf{p_0}(\mathsf{p_1}\mathfrak{s})(\mathsf{p_1}\mathfrak{r}y), \lambda z.\mathsf{p_0}\mathfrak{r}(\mathsf{p_1}(\mathsf{p_1}\mathfrak{s})z)\rangle, \beta, \mathsf{str}(\gamma, \mathsf{p_0}\mathfrak{s})],$$

which gives a realization of (Eq5).

□

Note. According to Theorem 1, to prove realizability of a theory $\mathrm{T_S}$, it's sufficient to construct realizing terms for non-logical axioms of $\mathrm{T_S}$. This is what we do in the following sections.

Convention about $(\mathbf{EET}+\mathbf{IG})\!\upharpoonright$. $(\mathbf{EET}+\mathbf{IG})\!\upharpoonright$ will be our default theory for reasoning in Explicit Mathematics.

Remark about non-wellfounded Set Theory. Since we are going to interpret Set Theory **CZF** with *Foundation* axiom, either full or restricted (cf. Sect. 3), we had to include inductive generator i into Definition 2.3. This is responsible for the fact that $\mathbf{IG}\!\upharpoonright$ is used already for interpreting the logic, Theorem 1.

If one is interested in variants of **CZF** with non-wellfounded sets, then $\mathbf{IG}\!\restriction$ may be unnecessary, but we may need induction on natural numbers instead. The exact situation depends on how much $\in$-induction one claims in the Set Theory. In the weakest cases the clause $t \dot{\subseteq} \mathsf{ig}t$ in the Definition 2.3 is superfluous, and both first-order logic (Th. 1) and *Extensionality* (Lemma 3.1 below) are realizable in $\mathbf{EET}\!\restriction$ alone.
We will see below (Sect. 3) that *every axiom of* **CZF***, except Foundation and Strong Collection, is realizable in* $(\mathbf{EET}+\mathbf{IG})\!\restriction$. For the same reason as above, in non-wellfounded setting **EET** alone could suffice. Full *Foundation* requires in addition full **IG** (Lemma 3.2), and *Strong Collection* requires **J** (Lemma 3.7).

To conclude this introductory section, we introduce the following two useful notions. For a given set-theoretic formula $F[C]$ Definition 2.9 provides an operation eq^C_F, which maps a bisimulator of trees α and β into a realizer of equivalence $F[\alpha] \leftrightarrow F[\beta]$ (Lemma 2.4). Then we define an *elementary realizability* $\underline{\mathrm{rn}}^{\mathcal{E}}$ for bounded formulas (Definition 2.10), and provide operations $\mathsf{p}_0\mathsf{eq}^{\mathcal{E}}$ and $\mathsf{p}_1\mathsf{eq}^{\mathcal{E}}$ which map standard realizes as defined on page 9 into elementary realizers and vice versa (Definition 2.11 and Lemma 2.5).

Definition 2.9 Equivalence operation, eq^C_F
For each formula $F \in \mathcal{L}_\in$ *and a free variable* $C \in \mathcal{L}_\in$ *by recursion on the built-up of* F *we define a term* $\mathsf{e}^C_F := \mathsf{e}^C_F[\mathfrak{r}]$ *in the following way:*

$$\mathsf{e}^C_F := \begin{cases}
\langle \lambda\mathfrak{x}.\mathfrak{x}, \lambda\mathfrak{y}.\mathfrak{y}\rangle & \text{if } F \text{ is } C=C,\ C\in C, \text{ or } A=B,\ A\in B \text{ with both } A,\ B \text{ not } C; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \lambda x.\mathsf{p}_0\mathfrak{x}(\mathsf{p}_1\mathfrak{r}x), \lambda y.\mathsf{p}_0\mathfrak{r}(\mathsf{p}_1\mathfrak{x}y)\rangle, \\ \lambda\mathfrak{y}.\langle \lambda x.\mathsf{p}_0\mathfrak{y}(\mathsf{p}_0\mathfrak{r}x), \lambda y.\mathsf{p}_1\mathfrak{r}(\mathsf{p}_1\mathfrak{y}y)\rangle\rangle\end{array} & \text{if } F \text{ is } C=B \text{ and } B \text{ is not } C; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \lambda x.\mathsf{p}_0\mathfrak{r}(\mathsf{p}_0\mathfrak{x}x), \lambda y.\mathsf{p}_1\mathfrak{x}(\mathsf{p}_1\mathfrak{r}y)\rangle, \\ \lambda\mathfrak{y}.\langle \lambda x.\mathsf{p}_1\mathfrak{r}(\mathsf{p}_0\mathfrak{y}x), \lambda y.\mathsf{p}_1\mathfrak{y}(\mathsf{p}_0\mathfrak{r}y)\rangle\rangle\end{array} & \text{if } F \text{ is } A=C \text{ and } A \text{ is not } C; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathfrak{x}, \\ \langle \lambda x.\mathsf{p}_0(\mathsf{p}_1\mathfrak{x})(\mathsf{p}_1\mathfrak{r}x), \lambda y.\mathsf{p}_0\mathfrak{r}(\mathsf{p}_1(\mathsf{p}_1\mathfrak{x})y)\rangle\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_0\mathfrak{y}, \\ \langle \lambda x.\mathsf{p}_0(\mathsf{p}_1\mathfrak{y})(\mathsf{p}_0\mathfrak{r}x), \lambda y.\mathsf{p}_1\mathfrak{r}(\mathsf{p}_1(\mathsf{p}_1\mathfrak{y})y)\rangle\rangle\rangle\end{array} & \text{if } F \text{ is } C\in B \text{ and } B \text{ is not } C; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathfrak{r}(\mathsf{p}_0\mathfrak{x}), \\ \langle \lambda x.\mathsf{p}_0\mathfrak{r}(\mathsf{p}_0(\mathsf{p}_1\mathfrak{x})x), \lambda y.\mathsf{p}_1(\mathsf{p}_1\mathfrak{x})(\mathsf{p}_1\mathfrak{r}y)\rangle\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_1\mathfrak{r}(\mathsf{p}_0\mathfrak{y}), \\ \langle \lambda x.\mathsf{p}_1\mathfrak{r}(\mathsf{p}_0(\mathsf{p}_1\mathfrak{y})x), \lambda y.\mathsf{p}_1(\mathsf{p}_1\mathfrak{y})(\mathsf{p}_0\mathfrak{r}y)\rangle\rangle\rangle\end{array} & \text{if } F \text{ is } A\in C \text{ and } A \text{ is not } C; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathsf{e}^C_{F_0}(\mathsf{p}_0\mathfrak{x}), \mathsf{p}_0\mathsf{e}^C_{F_1}(\mathsf{p}_1\mathfrak{x})\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_1\mathsf{e}^C_{F_0}(\mathsf{p}_0\mathfrak{y}), \mathsf{p}_1\mathsf{e}^C_{F_1}(\mathsf{p}_1\mathfrak{y})\rangle\rangle\end{array} & \text{if } F \text{ is } F_0 \wedge F_1; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathfrak{x}, \mathsf{p}_0\mathsf{d}_\mathsf{N}(\mathsf{p}_0\mathfrak{x}, 0, \mathsf{e}^C_{F_0}, \mathsf{e}^C_{F_1})(\mathsf{p}_1\mathfrak{x})\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_0\mathfrak{y}, \mathsf{p}_1\mathsf{d}_\mathsf{N}(\mathsf{p}_0\mathfrak{y}, 0, \mathsf{e}^C_{F_0}, \mathsf{e}^C_{F_1})(\mathsf{p}_1\mathfrak{y})\rangle\rangle\end{array} & \text{if } F \text{ is } F_0 \vee F_1; \\
\begin{array}{l}\langle \lambda\mathfrak{x}\lambda\mathfrak{z}.\mathsf{p}_0\mathsf{e}^C_{F_1}(\mathfrak{x}(\mathsf{p}_1\mathsf{e}^C_{F_0}\mathfrak{z})), \\ \lambda\mathfrak{y}\lambda\mathfrak{z}.\mathsf{p}_1\mathsf{e}^C_{F_1}(\mathfrak{y}(\mathsf{p}_0\mathsf{e}^C_{F_0}\mathfrak{z}))\rangle\end{array} & \text{if } F \text{ is } F_0 \to F_1; \\
\langle \lambda\mathfrak{x}\lambda k.\mathsf{p}_0\mathsf{e}^C_{G[K]}(\mathfrak{x}k), \lambda\mathfrak{y}\lambda k.\mathsf{p}_1\mathsf{e}^C_{G[K]}(\mathfrak{y}k)\rangle & \text{if } F \text{ is } \forall K G[K]; \\
\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathfrak{x}, \mathsf{p}_0\mathsf{e}^C_{G[K]}(\mathsf{p}_1\mathfrak{x})\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_0\mathfrak{y}, \mathsf{p}_1\mathsf{e}^C_{G[K]}(\mathsf{p}_1\mathfrak{y})\rangle\rangle\end{array} & \text{if } F \text{ is } \exists K G[K].
\end{cases} \tag{2.26}$$

We then define

$$\mathsf{eq}^C_F := \lambda\mathfrak{r}.\mathsf{e}^C_F[\mathfrak{r}]. \tag{2.27}$$

From this definition we have the following lemma:

Lemma 2.4 *If* $F[C] \in \mathcal{L}_\in$ *then*

$$\mathfrak{r}\,\underline{\mathrm{rn}}\,(A=B) \to \mathsf{eq}^C_F\mathfrak{r}\!\downarrow \underline{\mathrm{rn}}\,(F[A] \leftrightarrow F[B]).$$

Definition 2.10 $\mathfrak{r}$ elementarily realizes F, $\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F$
For each bounded formula $F \in \mathcal{L}_\in$ *we define an elementary formula* $(\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F) \in \mathcal{L}_{\mathbf{T_0}}$ *with a new free variable* $\mathfrak{r}$*. The definition is given by the table below:*

F	$\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F$
$A = B$	$BS[\mathfrak{r}, \alpha_A, \alpha_B]$
$A \in B$	$\mathsf{p}_0\mathfrak{r}\ \varepsilon\ \alpha_B \wedge \mathsf{ln}(\mathsf{p}_0\mathfrak{r}) = 1 \wedge BS[\mathsf{p}_1\mathfrak{r}, \alpha_A, \mathsf{str}(\alpha_B, \mathsf{p}_0\mathfrak{r})]$
$F_0 \wedge F_1$	$\mathsf{p}_0\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F_0 \wedge \mathsf{p}_1\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F_1$
$F_0 \vee F_1$	$\mathsf{p}_0\mathfrak{r}\ \varepsilon\ \mathsf{nat} \ \wedge \ \begin{array}{l}(\mathsf{p}_0\mathfrak{r} = 0 \to \mathsf{p}_1\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F_0) \ \wedge \\ (\mathsf{p}_0\mathfrak{r} \neq 0 \to \mathsf{p}_1\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} F_1)\end{array}$
$F_0 \to F_1$	$\forall \mathfrak{x}(\mathfrak{x}\,\underline{\mathrm{rn}}^{\mathcal{E}} F_0 \to \mathfrak{r}\mathfrak{x}\downarrow \wedge \ \mathfrak{r}\mathfrak{x}\,\underline{\mathrm{rn}}^{\mathcal{E}} F_1)$
$\forall X \in A G[X]$	$\forall x\ \varepsilon\ \alpha_A(\mathsf{ln} x = 1 \to \mathfrak{r}x\downarrow \wedge \ \mathfrak{r}x\,\underline{\mathrm{rn}}^{\mathcal{E}} G[\mathsf{str}(\alpha_A, x)])$
$\exists X \in A G[X]$	$\mathsf{p}_0\mathfrak{r}\ \varepsilon\ \alpha_A \wedge \mathsf{ln}(\mathsf{p}_0\mathfrak{r}) = 1 \wedge \mathsf{p}_1\mathfrak{r}\,\underline{\mathrm{rn}}^{\mathcal{E}} G[\mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{r})]$

Definition 2.11 Elementary equivalence operation, $\mathsf{eq}^{\mathcal{E}}_F$
For each bounded formula $F \in \mathcal{L}_\in$ by recursion on F we define a term $\mathsf{eq}^{\mathcal{E}}_F$ in the following way:

$$\mathsf{eq}^{\mathcal{E}}_F := \begin{cases} \langle \lambda\mathfrak{x}.\mathfrak{x}, \lambda\mathfrak{y}.\mathfrak{y}\rangle & \text{if } F \text{ is } A = B; \\ \langle \lambda\mathfrak{x}.\mathfrak{x}, \lambda\mathfrak{y}.\mathfrak{y}\rangle & \text{if } F \text{ is } A \in B; \\ \left.\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_{F_0}(\mathsf{p}_0\mathfrak{x}), \mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_{F_1}(\mathsf{p}_1\mathfrak{x})\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_{F_0}(\mathsf{p}_0\mathfrak{y}), \mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_{F_1}(\mathsf{p}_1\mathfrak{y})\rangle\rangle\end{array}\right\} & \text{if } F \text{ is } F_0 \wedge F_1; \\ \left.\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0\mathfrak{x}, \mathsf{p}_0(\mathsf{d_N}(\mathsf{p}_0\mathfrak{x}, 0, \mathsf{eq}^{\mathcal{E}}_{F_0}, \mathsf{eq}^{\mathcal{E}}_{F_1}))(\mathsf{p}_1\mathfrak{x})\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{p}_0\mathfrak{y}, \mathsf{p}_1(\mathsf{d_N}(\mathsf{p}_0\mathfrak{y}, 0, \mathsf{eq}^{\mathcal{E}}_{F_0}, \mathsf{eq}^{\mathcal{E}}_{F_1}))(\mathsf{p}_1\mathfrak{y})\rangle\rangle\end{array}\right\} & \text{if } F \text{ is } F_0 \vee F_1; \\ \left.\begin{array}{l}\langle \lambda\mathfrak{x}\lambda\mathfrak{z}.\mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_{F_1}(\mathfrak{x}(\mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_{F_0}\mathfrak{z})), \\ \lambda\mathfrak{y}\lambda\mathfrak{z}.\mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_{F_1}(\mathfrak{y}(\mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_{F_0}\mathfrak{z}))\rangle\end{array}\right\} & \text{if } F \text{ is } F_0 \to F_1; \\ \left.\begin{array}{l}\langle \lambda\mathfrak{x}\lambda x.\mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_{G[\mathsf{str}(\alpha_A, x)]}(\mathfrak{x}\mathsf{str}(\alpha_A, x)\langle x, \mathsf{ID}\rangle), \\ \lambda\mathfrak{y}\lambda\alpha\lambda\mathfrak{a}.\mathsf{p}_1(\mathsf{eq}^{X}_{G[X]}(\mathsf{p}_1\mathfrak{a}))(\mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_{G[\mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{a})]}(\mathfrak{y}(\mathsf{p}_0\mathfrak{a})))\rangle\end{array}\right\} & \text{if } F \text{ is } \forall X \in A G[X]; \\ \left.\begin{array}{l}\langle \lambda\mathfrak{x}.\langle \mathsf{p}_0(\mathsf{p}_0(\mathsf{p}_1\mathfrak{x})), \mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_{G[\mathsf{str}(\alpha_A, \mathsf{p}_0(\mathsf{p}_0(\mathsf{p}_1\mathfrak{x})))]}(\mathsf{p}_0(\mathsf{eq}^{X}_{G[X]}(\mathsf{p}_1(\mathsf{p}_0(\mathsf{p}_1\mathfrak{x}))))(\mathsf{p}_1(\mathsf{p}_1\mathfrak{x})))\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{y}), \langle\langle \mathsf{p}_0\mathfrak{y}, \mathsf{ID}\rangle, \mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_{G[\mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{y})]}(\mathsf{p}_1\mathfrak{y})\rangle\rangle\rangle\end{array}\right\} & \text{if } F \text{ is } \exists X \in A G[X]. \end{cases}$$

Lemma 2.5 Δ_0-lemma
If $F \in \mathcal{L}_\in$ is bounded then $\mathrm{FV}(\mathsf{eq}^{\mathcal{E}}_F) \subseteq \{\alpha_A \mid A \in \mathrm{FV}(F)\}$, $\mathsf{eq}^{\mathcal{E}}_F = \langle \mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_F, \mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_F\rangle$ *and the following holds:*

$$\mathfrak{x}\,\underline{\mathrm{rn}}\,F \to \mathsf{p}_0\mathsf{eq}^{\mathcal{E}}_F\mathfrak{x}\,\underline{\mathrm{rn}}^{\mathcal{E}} F \tag{2.28}$$

and

$$\mathfrak{y}\,\underline{\mathrm{rn}}^{\mathcal{E}} F \to \mathsf{p}_1\mathsf{eq}^{\mathcal{E}}_F\mathfrak{y}\,\underline{\mathrm{rn}}\,F. \tag{2.29}$$

Proof.
The condition on free variables and pairing follow directly from the definition. So do 2.28 and 2.29 as well; we show here only two interesting cases of bounded quantifiers.
By Definitions 2.6 and 2.10 we have:

$$\begin{aligned} \mathfrak{x}\,\underline{\mathrm{rn}}\,\forall X \in A G[X] \ &\equiv\ \forall\alpha\forall\mathfrak{a}(\mathrm{Set}[\alpha] \wedge \mathfrak{a}\,\underline{\mathrm{rn}}\,(\alpha \in \alpha_A) \to \mathfrak{x}\alpha\mathfrak{a}\downarrow \underline{\mathrm{rn}}\,G[\alpha]), \\ \mathfrak{y}\,\underline{\mathrm{rn}}^{\mathcal{E}}\forall X \in A G[X] \ &\equiv\ \forall x\ \varepsilon\ \alpha_A(\mathsf{ln} x = 1 \to \mathfrak{y}x\downarrow \underline{\mathrm{rn}}^{\mathcal{E}} G[\mathsf{str}(\alpha_A, x)]); \end{aligned}$$

$$\begin{aligned} \mathfrak{x}\,\underline{\mathrm{rn}}\,\exists X \in A G[X] \ &\equiv\ \mathrm{Set}[\mathsf{p}_0\mathfrak{x}] \wedge \mathsf{p}_0(\mathsf{p}_1\mathfrak{x})\,\underline{\mathrm{rn}}\,(\mathsf{p}_0\mathfrak{x} \in \alpha_A) \wedge \mathsf{p}_1(\mathsf{p}_1\mathfrak{x})\,\underline{\mathrm{rn}}\,G[\mathsf{p}_0\mathfrak{x}], \\ \mathfrak{y}\,\underline{\mathrm{rn}}^{\mathcal{E}}\exists X \in A G[X] \ &\equiv\ \mathsf{p}_0\mathfrak{y}\ \varepsilon\ \alpha_A \wedge \mathsf{ln}(\mathsf{p}_0\mathfrak{y}) = 1 \wedge \mathsf{p}_1\mathfrak{y}\,\underline{\mathrm{rn}}^{\mathcal{E}} G[\mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{y})]. \end{aligned}$$

Equations 2.28 and 2.29 are verified using Lemmas 2.1, 2.4 and induction hypothesis.

□

In addition, we define "mixtures" of $\underline{\mathrm{rn}}$ and $\underline{\mathrm{rn}}^{\mathcal{E}}$: $\underline{\mathrm{rn}}^{\forall}$, $\underline{\mathrm{rn}}^{\exists}$, $\mathsf{eq}^{\forall}$ and $\mathsf{eq}^{\exists}$. They simplify treatment of bounded quantifiers, even without the assumption that the rest of formula is bounded.

Definition 2.12 $\underline{\text{rn}}^{\forall}$, $\underline{\text{rn}}^{\exists}$, $\mathsf{eq}^{\forall}$, $\mathsf{eq}^{\exists}$

1. $\mathfrak{r}\,\underline{\text{rn}}^{\forall}\forall X\in AG[X]$ *is defined as* $\forall x\,\varepsilon\,\alpha_A(\mathsf{ln}x = 1 \to \mathfrak{r}x\downarrow \wedge\ \mathfrak{r}x\,\underline{\text{rn}}\,G[\mathsf{str}(\alpha_A, x)])$.

2. $\mathfrak{r}\,\underline{\text{rn}}^{\exists}\exists X\in AG[X]$ *is defined as* $\mathsf{p}_0\mathfrak{r}\,\varepsilon\,\alpha_A \wedge \mathsf{ln}(\mathsf{p}_0\mathfrak{r}) = 1 \wedge \mathsf{p}_1\mathfrak{r}\,\underline{\text{rn}}\,G[\mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{r})]$.

3. $$\mathsf{eq}^{\forall}_{\forall X\in AG[X]} := \begin{cases}\langle \lambda\mathfrak{r}\lambda x.\mathfrak{r}\mathsf{str}(\alpha_A, x)\langle x, \mathsf{ID}\rangle, \\ \lambda\mathfrak{y}\lambda\alpha\lambda\mathfrak{a}.\mathsf{p}_1(\mathsf{eq}^X_{G[X]}(\mathsf{p}_1\mathfrak{a}))(\mathfrak{y}(\mathsf{p}_0\mathfrak{a}))\rangle\end{cases}.$$

4. $$\mathsf{eq}^{\exists}_{\exists X\in AG[X]} := \begin{cases}\langle \lambda\mathfrak{r}.\langle \mathsf{p}_0(\mathsf{p}_0(\mathsf{p}_1\mathfrak{r})), \mathsf{p}_0(\mathsf{eq}^X_{G[X]}(\mathsf{p}_1(\mathsf{p}_0(\mathsf{p}_1\mathfrak{r}))))(\mathsf{p}_1(\mathsf{p}_1\mathfrak{r}))\rangle, \\ \lambda\mathfrak{y}.\langle \mathsf{str}(\alpha_A, \mathsf{p}_0\mathfrak{y}), \langle\langle \mathsf{p}_0\mathfrak{y}, \mathsf{ID}\rangle, \mathsf{p}_1\mathfrak{y}\rangle\rangle\rangle\end{cases}.$$

This definition invokes an obvious lemma:

Lemma 2.6 $\forall\exists$-lemma

1. *If* $F \in \mathcal{L}_\in$ *is* $\forall X\in AG[X]$ *then* $\mathrm{FV}(\mathsf{eq}^{\forall}_F) = \{\alpha_A\}$, $\mathsf{eq}^{\forall}_F = \langle \mathsf{p}_0\mathsf{eq}^{\forall}_F, \mathsf{p}_1\mathsf{eq}^{\forall}_F\rangle$ *and the following holds:*
$$(\mathfrak{r}\,\underline{\text{rn}}\,F \to \mathsf{p}_0\mathsf{eq}^{\forall}_F\mathfrak{r}\,\underline{\text{rn}}^{\forall}F) \ \wedge\ (\mathfrak{y}\,\underline{\text{rn}}^{\forall}F \to \mathsf{p}_1\mathsf{eq}^{\forall}_F\mathfrak{y}\,\underline{\text{rn}}\,F). \tag{2.30}$$

2. *If* $F \in \mathcal{L}_\in$ *is* $\exists X\in AG[X]$ *then* $\mathrm{FV}(\mathsf{eq}^{\forall}_F) = \{\alpha_A\}$, $\mathsf{eq}^{\exists}_F = \langle \mathsf{p}_0\mathsf{eq}^{\exists}_F, \mathsf{p}_1\mathsf{eq}^{\exists}_F\rangle$ *and the following holds:*
$$(\mathfrak{r}\,\underline{\text{rn}}\,F \to \mathsf{p}_0\mathsf{eq}^{\exists}_F\mathfrak{r}\,\underline{\text{rn}}^{\exists}F) \ \wedge\ (\mathfrak{y}\,\underline{\text{rn}}^{\exists}F \to \mathsf{p}_1\mathsf{eq}^{\exists}_F\mathfrak{y}\,\underline{\text{rn}}\,F). \tag{2.31}$$

Remark. We will often mix freely all four kinds of realizations introduced in this section: $\underline{\text{rn}}$, $\underline{\text{rn}}^{\mathcal{E}}$, $\underline{\text{rn}}^{\forall}$ and $\underline{\text{rn}}^{\exists}$, having in mind that the realizers can be effectively mapped into each other.

3 Realizing CZF in $\mathbf{T_0}$

The language of **CZF** is $\mathcal{L}_\in$. The logic is *intuitionistic first-order with equality.*
In the remainder of this paper we will use the following abbreviations:

$W = \{U, V\}$	for	$U \in W \wedge V \in W \wedge \forall X\in W(X = U \vee X = V)$,
$W = \langle U, V\rangle$	-"-	$U \in W \wedge \exists X\in W(X = \{U, V\}) \wedge \forall X\in W(X = U \vee X = \{U, V\})$,
$\langle U, V\rangle \in R$	-"-	$\exists W\in R(W = \langle U, V\rangle)$,
$R \subseteq A \times B$	-"-	$\forall W\in R\exists U\in A\exists V\in B(W = \langle U, V\rangle)$,
$\mathbf{mv}[R, A, B]$	-"-	$R \subseteq A \times B \wedge \forall U\in A\exists V\in B(\langle U, V\rangle \in R)$,
$\mathrm{Full}[C, A, B]$	-"-	$\forall W\in C\mathbf{mv}[W, A, B] \wedge \forall R(\mathbf{mv}[R, A, B] \to \exists S\in C\forall W\in S(W \in R))$.

Note that all these formulas, except $\mathrm{Full}[C, A, B]$, are bounded.

CZF has the following non-logical axioms:

Extensionality:	$\forall X\forall Y(\forall Z\in X(Z \in Y) \wedge \forall Z\in Y(Z \in X) \to X = Y)$
Foundation:	$\forall X(\forall Y\in XG[Y] \to G[X]) \to \forall XG[X]$ for all formulas $G[X]$
Pair:	$\forall X\forall Y\exists Z\forall U(U \in Z \leftrightarrow U = X \vee U = Y)$
Union:	$\forall X\exists Y\forall U(U \in Y \leftrightarrow \exists V\in X(U \in V))$
Infinity:	$\exists X(\exists Z(Z \in X) \wedge \forall Y\in X\exists Z \in X(Y \in Z))$
Bounded Separation:	$\forall X\exists Y\forall U(U \in Y \leftrightarrow U \in X \wedge F[U])$ for all bounded formulas $F[U]$
Strong Collection:	$\forall X(\forall U\in X\exists YG[U, Y] \to \exists W(\forall U\in X\exists Y\in WG[U, Y] \wedge \forall Y\in W\exists U\in XG[U, Y]))$ for all formulas $G[U, Y]$
Fullness:	$\forall X\forall Y\exists Z\mathrm{Full}[Z, X, Y]$

Alternatively, instead of *Fullness*, we could take the following schema:

Subset Collection:

$$\forall X \forall X' \exists Z (\forall U \in X \exists Y \in X' G[U,Y] \to \exists W \in Z (\forall U \in X \exists Y \in W G[U,Y] \wedge \forall Y \in W \exists U \in X G[U,Y])) \quad \text{for all formulas } G[U,Y]$$

As shown in [RGP98] (Proposition 2.3(i)), *Fullness* $\leftrightarrow$ *Subset Collection* on the basis of remaining axioms of **CZF**. Also, similarly to [Myh75], Appendix A, *Bounded Separation* schema can be replaced by a finite number of its special cases.

We will also consider a theory **CZF**$\restriction$, which is **CZF** where the formula $G[X]$ in the *Foundation* schema must be of the form $X \in U$.

It's convenient to separate axioms of **CZF** into two groups: *axioms describing properties of sets*, which are *Extensionality* and *Foundation*, and *set-existence axioms*, which are all the rest. Giving realizations for axioms of the first group boils down to verifying that Definition 2.6 indeed satisfies those properties. More specifically, we have to verify that the notion of bisimulation is extensional, and inductive generator i included into the Definition 2.3 provides for $\in$-induction for every formula. Set-existence axioms call for an explicit construction of appropriate trees, e.g. to realize *Pair* we need to show how to construct Z from X and Y, and verify that our construction is correct, i.e. to exhibit a realizer of $\forall U(U \in Z \leftrightarrow U = X \vee U = Y)$.

Lemma 3.1 *Extensionality*
Extensionality axiom is realizable.

Proof. Given $\mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta]$ and $\mathfrak{r}\, \underline{\mathrm{rn}}^{\mathcal{E}} (\forall Z \in X (Z \in Y) \wedge \forall Z \in Y (Z \in X))$, we need to build a bisimulator $\mathfrak{s}$ for α and β.
Assume $x \,\varepsilon\, \alpha$.
If $\mathsf{ln} x = 0$, we can set

$$\mathsf{p_0} \mathfrak{s} x := \mathsf{nil}. \tag{3.1}$$

Assume $\mathsf{ln} x \neq 0$. Then $\mathsf{head} x \,\varepsilon\, \alpha$ and $\mathsf{ln}(\mathsf{head} x) = 1$. Then

$$\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)\!\downarrow \underline{\mathrm{rn}}^{\mathcal{E}}(\mathsf{str}(\alpha, \mathsf{head}x) \in \beta), \tag{3.2}$$

which reads

$$\begin{gathered}\mathsf{p_0}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)) \,\varepsilon\, \beta \wedge \mathsf{ln}(\mathsf{p_0}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x))) = 1 \wedge \\ BS[\mathsf{p_1}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)), \mathsf{str}(\alpha, \mathsf{head}x), \mathsf{str}(\beta, \mathsf{p_0}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)))].\end{gathered} \tag{3.3}$$

We have $\mathsf{bodyt}x \,\varepsilon\, \mathsf{str}(\alpha, \mathsf{head}x)$, so by 3.3

$$\begin{gathered}\mathsf{p_0}(\mathsf{p_1}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)))(\mathsf{bodyt}x) \,\varepsilon\, \mathsf{str}(\beta, \mathsf{p_0}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x))), \\ \mathsf{ln}(\mathsf{p_0}(\mathsf{p_1}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)))(\mathsf{bodyt}x)) = \mathsf{ln}(\mathsf{bodyt}x).\end{gathered} \tag{3.4}$$

Then we set

$$\begin{gathered}\mathsf{p_0}\mathfrak{s}x := \mathsf{p_0}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)) * \mathsf{p_0}(\mathsf{p_1}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)))(\mathsf{bodyt}x) \,\varepsilon\, \beta, \\ \mathsf{ln}(\mathsf{p_0}\mathfrak{s}x) = \mathsf{ln}x.\end{gathered} \tag{3.5}$$

Therefore, from 3.1 and 3.5, the first component of $\mathfrak{s}$ is set to be

$$\mathsf{p_0}\mathfrak{s} := \lambda x.\mathsf{d_N}\left(\mathsf{ln}x, 0, \mathsf{nil}, \mathsf{p_0}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)) * \mathsf{p_0}(\mathsf{p_1}(\mathsf{p_0}\mathfrak{r}(\mathsf{head}x)))(\mathsf{bodyt}x)\right). \tag{3.6}$$

Symmetrically we construct

$$\mathsf{p_1}\mathfrak{s} := \lambda y.\mathsf{d_N}\left(\mathsf{ln}y, 0, \mathsf{nil}, \mathsf{p_0}(\mathsf{p_1}\mathfrak{r}(\mathsf{head}y)) * \mathsf{p_0}(\mathsf{p_1}(\mathsf{p_1}\mathfrak{r}(\mathsf{head}y)))(\mathsf{bodyt}y)\right). \tag{3.7}$$

and then set

$$\mathfrak{s} := \langle \mathsf{p_0}\mathfrak{s}, \mathsf{p_1}\mathfrak{s} \rangle. \tag{3.8}$$

According to this construction, $BS[\mathfrak{s}, \alpha, \beta]$ follows easily.

□

Lemma 3.2 *Foundation*
a) Every instance of Foundation is realizable in **EET + IG***;*
b) Restricted Foundation is realizable in (**EET + IG**)$\restriction$.

Proof. a) Assume $\mathfrak{r}\ \underline{\mathrm{rn}}\ \forall X(\forall Y\in X G[Y]\to G[X])$, which reads

$$\begin{aligned}&\forall\alpha\forall\mathfrak{v}\,\big(\mathrm{Set}[\alpha]\wedge\mathfrak{v}\ \underline{\mathrm{rn}}\ (\forall Y\in\alpha G[Y])\to\mathfrak{r}\alpha\mathfrak{v}\!\downarrow\underline{\mathrm{rn}}\ G[\alpha]\big)\equiv\\&\forall\alpha\forall\mathfrak{v}\,\big(\mathrm{Set}[\alpha]\wedge\forall\beta\forall\mathfrak{y}(\mathrm{Set}[\beta]\wedge\mathfrak{y}\ \underline{\mathrm{rn}}\ (\beta\in\alpha)\to\mathfrak{v}\beta\mathfrak{y}\!\downarrow\underline{\mathrm{rn}}\ G[\beta])\to\mathfrak{r}\alpha\mathfrak{v}\!\downarrow\underline{\mathrm{rn}}\ G[\alpha]\big)\equiv\\&\forall\alpha\forall\mathfrak{v}\,\big(\mathrm{Set}[\alpha]\wedge\forall\beta\forall\mathfrak{y}(\mathrm{Set}[\beta]\wedge\mathsf{p}_0\mathfrak{y}\,\varepsilon\,\alpha\wedge\mathsf{ln}(\mathsf{p}_0\mathfrak{y})=1\wedge BS[\mathsf{p}_1\mathfrak{y},\beta,\mathsf{str}(\alpha,\mathsf{p}_0\mathfrak{y})]\to\\&\qquad\mathfrak{v}\beta\mathfrak{y}\!\downarrow\underline{\mathrm{rn}}\ G[\beta])\to\mathfrak{r}\alpha\mathfrak{v}\!\downarrow\underline{\mathrm{rn}}\ G[\alpha]\big)\,.\end{aligned}\tag{3.9}$$

Assume also Set$[\gamma]$. Instantiating $\alpha:=\mathsf{str}(\gamma,z)$ into 3.9, we have

$$\begin{aligned}&\forall z\,\varepsilon\,\gamma\forall\mathfrak{v}\,\big(\forall\beta\forall\mathfrak{y}(\mathrm{Set}[\beta]\wedge\mathsf{p}_0\mathfrak{y}\,\varepsilon\,\mathsf{str}(\gamma,z)\wedge\mathsf{ln}(\mathsf{p}_0\mathfrak{y})=1\wedge BS[\mathsf{p}_1\mathfrak{y},\beta,\mathsf{str}(\mathsf{str}(\gamma,z),\mathsf{p}_0\mathfrak{y})]\to\\&\qquad\mathfrak{v}\beta\mathfrak{y}\!\downarrow\underline{\mathrm{rn}}\ G[\beta])\to\mathfrak{r}\mathsf{str}(\gamma,z)\mathfrak{v}\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,z)]\big)\,.\end{aligned}\tag{3.10}$$

By recursion theorem for a function $f:=\lambda c\lambda z.\mathfrak{r}\mathsf{str}(\gamma,z)\lambda\beta\lambda\mathfrak{y}.\mathsf{p}_1(\mathsf{eq}^C_{G[C]}(\mathsf{p}_1\mathfrak{y}))(\mathsf{R}(z*\mathsf{p}_0\mathfrak{y}))$ there exists a term $\mathsf{R}:=\mathsf{rec}f$ such that

$$\mathsf{R}z\simeq\mathfrak{r}\mathsf{str}(\gamma,z)\lambda\beta\lambda\mathfrak{y}.\mathsf{p}_1(\mathsf{eq}^C_{G[C]}(\mathsf{p}_1\mathfrak{y}))(\mathsf{R}(z*\mathsf{p}_0\mathfrak{y})).\tag{3.11}$$

We want to prove $\mathrm{Prog}_{\sqsupset}(\gamma,\mathsf{R}z\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,z)])$, i.e.

$$z\,\varepsilon\,\gamma\to\big(\forall u\,\varepsilon\,\gamma(u\sqsupset z\to\mathsf{R}u\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,u)])\to\mathsf{R}z\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,z)]\big)\,.\tag{3.12}$$

Assume $z\,\varepsilon\,\gamma\wedge\forall u\,\varepsilon\,\gamma(u\sqsupset z\to\mathsf{R}u\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,u)])$. If $\mathrm{Set}[\beta]\wedge\mathfrak{y}\ \underline{\mathrm{rn}}\ (\beta\in\mathsf{str}(\gamma,z))$, then

$$z*\mathsf{p}_0\mathfrak{y}\,\varepsilon\,\gamma,\ \ z*\mathsf{p}_0\mathfrak{y}\sqsupset z,\tag{3.13}$$

and by assumption

$$\mathsf{R}(z*\mathsf{p}_0\mathfrak{y})\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,z*\mathsf{p}_0\mathfrak{y})].\tag{3.14}$$

From Definition 2.6 we also have in this case $\mathsf{p}_1\mathfrak{y}\ \underline{\mathrm{rn}}\ (\beta=\mathsf{str}(\mathsf{str}(\gamma,z),\mathsf{p}_0\mathfrak{y}))$, which together with

$$\mathsf{ID}\ \underline{\mathrm{rn}}\ (\mathsf{str}(\mathsf{str}(\gamma,z),\mathsf{p}_0\mathfrak{y})=\mathsf{str}(\gamma,z*\mathsf{p}_0\mathfrak{y}))\tag{3.15}$$

by Lemma 2.2c gives $\mathsf{p}_1\mathfrak{y}\ \underline{\mathrm{rn}}\ (\beta=\mathsf{str}(\gamma,z*\mathsf{p}_0\mathfrak{y}))$ and by Lemma 2.4

$$\mathsf{p}_1(\mathsf{eq}^C_{G[C]}(\mathsf{p}_1\mathfrak{y}))(\mathsf{R}(z*\mathsf{p}_0\mathfrak{y}))\!\downarrow\underline{\mathrm{rn}}\ G[\beta].\tag{3.16}$$

Therefore for the operation $\mathfrak{v}:=\mathfrak{v}[z]:=\lambda\beta\lambda\mathfrak{y}.\mathsf{p}_1(\mathsf{eq}^C_{G[C]}(\mathsf{p}_1\mathfrak{y}))(\mathsf{R}(z*\mathsf{p}_0\mathfrak{y}))$ by 3.10 we have $\mathfrak{r}\mathsf{str}(\gamma,z)\mathfrak{v}\!\downarrow\underline{\mathrm{rn}}\,G[\mathsf{str}(\gamma,z)]$, i.e. $\mathfrak{r}\mathsf{str}(\gamma,z)\lambda\beta\lambda\mathfrak{y}.\mathsf{p}_1(\mathsf{eq}^C_{G[C]}(\mathsf{p}_1\mathfrak{y}))(\mathsf{R}(z*\mathsf{p}_0\mathfrak{y}))\!\downarrow\underline{\mathrm{rn}}\,G[\mathsf{str}(\gamma,z)]$. From this fact and equation 3.11 we obtain 3.12.
By **IG** we obtain

$$\forall z\,\varepsilon\,\gamma(\mathsf{R}z\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,z)]).\tag{3.17}$$

Letting $z:=\mathsf{nil}$, we have

$$\mathsf{Rnil}\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,\mathsf{nil})],\tag{3.18}$$

which, together with the fact

$$\mathsf{ID}\ \underline{\mathrm{rn}}\ (\gamma=\mathsf{str}(\gamma,\mathsf{nil})),\tag{3.19}$$

by Lemma 2.4 yields

$$\mathsf{p}_1(\mathsf{eq}^C_{G[C]}\mathsf{ID})(\mathsf{Rnil})\!\downarrow\underline{\mathrm{rn}}\ G[\gamma].\tag{3.20}$$

This shows that an operation $\lambda\mathfrak{r}\lambda\gamma.\mathsf{p}_1(\mathsf{eq}^C_{G[C]}\mathsf{ID})(\mathsf{Rnil})$ is a realization of an instance of *Foundation*

$$\forall X(\forall Y\in X G[Y]\to G[X])\to\forall X G[X].$$

b) If $G[X]$ is $X\in U$, then it's enough to observe that

$$\begin{aligned}\mathsf{R}z\!\downarrow\underline{\mathrm{rn}}\ G[\mathsf{str}(\gamma,z)]\ &\equiv\ \mathsf{R}z\!\downarrow\underline{\mathrm{rn}}\ (\mathsf{str}(\gamma,z)\in\alpha_U)\ \equiv\\ \mathsf{p}_0(\mathsf{R}z)\,\varepsilon\,\alpha_U\wedge\mathsf{ln}(\mathsf{p}_0(\mathsf{R}z))=1&\wedge BS[\mathsf{p}_1(\mathsf{R}z),\mathsf{str}(\gamma,z),\mathsf{str}(\alpha_U,\mathsf{p}_0(\mathsf{R}z))]\end{aligned}\tag{3.21}$$

is elementary in α_U, γ, and therefore can be written as $z \varepsilon t[\alpha_U, \gamma]$ for some name t.

□

Rest of the axioms of **CZF** are set-existence axioms. To realize those, one has to explicitly construct a wellfounded tree from a given data. Correctness will follow routinely from the construction, and mostly will be left to the reader.

Definition 3.1 Pairings pt, opt and projections pt_0, pt_1

1. *By* **ECA** *we define a name* $p[\alpha, \beta]$ *so that*

$$c \,\varepsilon\, p[\alpha, \beta] \leftrightarrow c \,\varepsilon\, \mathsf{seq} \wedge P[\alpha, \beta, c], \tag{3.22}$$

where $P[\alpha, \beta, c]$ *is a formula*

$$\mathsf{ln}c = 0 \;\vee\; (\mathsf{ln}c \neq 0 \wedge \mathsf{ct}c = 0 \wedge \mathsf{body}\mathsf{t}c \,\varepsilon\, \alpha) \;\vee\; (\mathsf{ln}c \neq 0 \wedge \mathsf{ct}c = 1 \wedge \mathsf{body}\mathsf{t}c \,\varepsilon\, \beta). \tag{3.23}$$

pt *is defined as* $\lambda\alpha\lambda\beta.p[\alpha, \beta]$, opt *is* $\lambda\alpha\lambda\beta.p[\alpha, p[\alpha, \beta]]$.

2. *Again by* **ECA** *we define names* $p_i[\alpha]$, $i = 0, 1$, *so that*

$$c \,\varepsilon\, p_i[\alpha] \leftrightarrow c \,\varepsilon\, \mathsf{seq} \wedge P_i[\alpha, c], \tag{3.24}$$

where $P_0[\alpha, c]$ *is a formula*

$$\mathsf{sg}0 * c \,\varepsilon\, \alpha. \tag{3.25}$$

and $P_1[\alpha, c]$ *is a formula*

$$\langle 1, \langle \mathsf{sg}1, 1\rangle\rangle * c \,\varepsilon\, \alpha. \tag{3.26}$$

pt_0 *and* pt_1 *are defined as* $\lambda\alpha.p_0[\alpha]$ *and* $\lambda\alpha.p_1[\alpha]$, *resp.*

These operations from w.-f. trees α and β give a *pairing tree* $\mathsf{pt}\alpha\beta$ and an *ordered pairing tree* $\mathsf{opt}\alpha\beta$, and pt_0 and pt_1 are projection-operations corresponding to opt. Immediately from the definitions we have the following facts.

Proposition 3.1
a) $\mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta] \rightarrow \mathrm{Set}[\mathsf{pt}\alpha\beta] \wedge \mathrm{Set}[\mathsf{opt}\alpha\beta] \wedge \mathrm{Set}[\mathsf{pt}_0\alpha] \wedge \mathrm{Set}[\mathsf{pt}_1\alpha]$;
b) there are elementary realizers for formulas $\mathsf{pt}\alpha\beta = \{\alpha, \beta\}$ *and* $\mathsf{opt}\alpha\beta = \langle\alpha, \beta\rangle$;
c) $\mathsf{ID}\ \underline{\mathrm{rn}}\ (\mathsf{pt}_0(\mathsf{opt}\alpha\beta) = \alpha) \wedge \mathsf{ID}\ \underline{\mathrm{rn}}\ (\mathsf{pt}_1(\mathsf{opt}\alpha\beta) = \beta)$.

Proof. a) follows from 3.22–3.26.
b): By 3.23 $\mathsf{t} := \langle\langle\langle \mathsf{sg}0, \mathsf{ID}\rangle, \langle \mathsf{sg}1, \mathsf{ID}\rangle\rangle, \lambda x.\langle \mathsf{ct}x, \mathsf{ID}\rangle\rangle$ is an elementary realizer of $\mathsf{pt}\alpha\beta = \{\alpha, \beta\}$ and
$\mathsf{u} := \langle\langle\langle \mathsf{sg}0, \mathsf{ID}\rangle, \langle \mathsf{sg}1, \mathsf{t}\rangle\rangle, \lambda x.\langle \mathsf{ct}x, \mathsf{d_N}(\mathsf{ct}x, 0, \mathsf{ID}, \mathsf{t})\rangle\rangle$ is an elementary realizer of $\mathsf{opt}\alpha\beta = \langle\alpha, \beta\rangle$.
c): This is just the statement that $\mathsf{pt}_0(\mathsf{opt}\alpha\beta) \doteq \alpha$ and $\mathsf{pt}_1(\mathsf{opt}\alpha\beta) \doteq \beta$, which again follows from 3.22–3.26.

□

Lemma 3.3 *Pair*
Axiom Pair is realizable.

Proof. pt operation gives a realization of *Pair*:

$$\forall X \forall Y \exists Z \forall U (U \in Z \leftrightarrow U = X \vee U = Y) :$$

given $\mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta]$, take Z to be $\mathsf{pt}\alpha\beta$. realization of $\forall U(U \in \mathsf{pt}\alpha\beta \leftrightarrow U = \alpha \vee U = \beta)$ follows from the construction 3.22 of $p[\alpha, \beta]$.

□

Lemma 3.4 *Union*
Axiom Union is realizable.

Proof. Assume Set$[\alpha]$. Consider the following formula $U[\alpha, c]$:

$$\begin{aligned} &\mathsf{ln}c = 0 \ \vee \ \big(\mathsf{ln}c \neq 0 \wedge \mathsf{ct}c = \langle \mathsf{p}_0(\mathsf{ct}c), \mathsf{p}_1(\mathsf{ct}c)\rangle \ \wedge \\ &(\mathsf{sg}(\mathsf{p}_0(\mathsf{ct}c)) * \mathsf{sg}(\mathsf{p}_1(\mathsf{ct}c))) * \mathsf{body}\mathsf{t}c \,\varepsilon\, \alpha\big) \ . \end{aligned} \tag{3.27}$$

By **ECA** we define a name $u[\alpha]$ so that

$$c \,\varepsilon\, u[\alpha] \leftrightarrow c \,\varepsilon\, \mathsf{seq} \wedge U[\alpha, c] \tag{3.28}$$

($u[\alpha]$ "squeezes together" every first two members of α). Note that Set$[u[\alpha]]$. This $u[\alpha]$ serves as a witness for Y in the axiom

$$\forall X \exists Y \forall U (U \in Y \leftrightarrow \exists V \in X (U \in V)).$$

□

Lemma 3.5 *Infinity*
Axiom Infinity is realizable.

Proof. Infinite tree is constructed as follows. By primitive recursion a sequence $\mathsf{sq}^0 n$ of 0's of length n is defined by:

$$\begin{cases} \mathsf{sq}^0 0 = \mathsf{nil}, \\ \mathsf{sq}^0(n') = \langle 1, \langle \mathsf{sq}^0 n, 0\rangle\rangle. \end{cases} \tag{3.29}$$

Now, we index each such sequence by its length: by **ECA** there is a name inf s.t.

$$b \,\varepsilon\, \mathsf{inf} \leftrightarrow b = \mathsf{nil} \vee (b \,\varepsilon\, \mathsf{seq} \wedge \mathsf{ln}b \neq 0 \wedge \mathsf{ct}b \,\varepsilon\, \mathsf{nat} \wedge \mathsf{body}\mathsf{t}b = \mathsf{sq}^0(\mathsf{ct}b)). \tag{3.30}$$

□

Lemma 3.6 *Bounded Separation*
Every instance of Bounded Separation is realizable.

Proof. Assume a formula F to be bounded. Assume Set$[\alpha]$. Consider the following formula $S[\alpha, c]$:

$$\begin{aligned} &\mathsf{ln}c = 0 \ \bigvee \\ &\mathsf{ln}c \neq 0 \wedge c = \mathsf{sm}(\mathsf{sm}_0 c, \mathsf{sm}_1 c) \wedge \mathsf{sm}_0 c \,\varepsilon\, \alpha \wedge \mathsf{sm}_1 c \ \underline{\mathrm{rn}}^{\mathcal{E}} F[\mathsf{str}(\alpha, \mathsf{head}(\mathsf{sm}_0 c))]. \end{aligned} \tag{3.31}$$

By **ECA** there exists a name $s[\alpha]$ such that

$$c \,\varepsilon\, s[\alpha] \leftrightarrow c \,\varepsilon\, \mathsf{seq} \wedge S[\alpha, c]. \tag{3.32}$$

Obviously, Set$[s[\alpha]]$. This $s[\alpha]$ is taken as a name for Y in the instance of *Bounded Separation:*

$$\forall X \exists Y \forall U (U \in Y \leftrightarrow U \in X \wedge F[U]).$$

Also, by 3.31,

$$c \,\varepsilon\, s[\alpha] \wedge \mathsf{ln}c = 1 \to \mathsf{sm}_0 c \,\varepsilon\, \alpha \wedge \mathsf{ln}(\mathsf{sm}_0 c) = 1 \wedge \mathsf{sm}_1 c \ \underline{\mathrm{rn}}^{\mathcal{E}} F[\mathsf{str}(\alpha, \mathsf{sm}_0 c)] \tag{3.33}$$

and

$$d \,\varepsilon\, \alpha \wedge \mathsf{ln}d = 1 \wedge \mathfrak{r} \ \underline{\mathrm{rn}}^{\mathcal{E}} F[\mathsf{str}(\alpha, d)] \to \mathsf{sm}(d, \mathfrak{r}) \,\varepsilon\, s[\alpha] \wedge \mathsf{ln}(\mathsf{sm}(d, \mathfrak{r})) = 1. \tag{3.34}$$

This is sufficient to build a realizer of $\forall U (U \in s[\alpha] \leftrightarrow U \in \alpha \wedge F[U])$.

□

Lemma 3.7 *Strong Collection*
Every instance of the axiom Strong Collection is realizable.

Proof. Let a formula $G[U,Y] \in \mathcal{L}_\in$ be given. Assume $\mathrm{Set}[\alpha]$. Assume also $\mathfrak{r}\ \underline{\mathrm{rn}}\ \forall U \in \alpha \exists Y G[U,Y]$, which reads

$$\forall v \forall \mathfrak{u}(\mathrm{Set}[v] \wedge \mathfrak{u}\ \underline{\mathrm{rn}}\ (v \in \alpha) \to \mathrm{Set}[\mathsf{p}_0(\mathfrak{r}v\mathfrak{u})] \wedge \mathsf{p}_1(\mathfrak{r}v\mathfrak{u})\ \underline{\mathrm{rn}}\ G[v, \mathsf{p}_0(\mathfrak{r}v\mathfrak{u})]). \tag{3.35}$$

In particular, taking $v := \mathsf{str}(\alpha, z)$, $\mathfrak{u} := \langle z, \mathsf{ID}\rangle$, for $z \,\varepsilon\, \alpha$, $\mathsf{ln} z = 1$, and denoting $\mathfrak{f} := \mathfrak{f}[\mathfrak{r}, \alpha, z] := \mathfrak{r}\mathsf{str}(\alpha, z)\mathfrak{u}$, we have

$$\forall z \,\varepsilon\, \alpha(\mathsf{ln} z = 1 \to \mathrm{Set}[\mathsf{p}_0\mathfrak{f}] \wedge \mathsf{p}_1\mathfrak{f}\ \underline{\mathrm{rn}}\ G[\mathsf{str}(\alpha, z), \mathsf{p}_0\mathfrak{f}]). \tag{3.36}$$

Now we index each of the trees $\mathsf{p}_0\mathfrak{f}[\mathfrak{r}, \alpha, z]$ by z and, using **J**, collect them together into a single tree. Formally, consider the following formula $SC[\mathfrak{r}, \alpha, c]$:

$$\mathsf{ln} c = 0 \ \vee\ \left(\mathsf{ln} c \neq 0 \wedge \mathsf{head} c \,\varepsilon\, \alpha \wedge \mathsf{bodyt} c \,\varepsilon\, \mathsf{p}_0\mathfrak{f}[\mathfrak{r}, \alpha, \mathsf{head} c]\right). \tag{3.37}$$

Using 3.36, by **J** and **ECA** there exists a name $sc[\mathfrak{r}, \alpha]$ such that

$$c \,\varepsilon\, sc[\mathfrak{r}, \alpha] \leftrightarrow c \,\varepsilon\, \mathsf{seq} \wedge SC[\mathfrak{r}, \alpha, c]. \tag{3.38}$$

By construction 3.37, $\mathrm{Set}[sc[\mathfrak{r}, \alpha]]$. This $sc[\mathfrak{r}, \alpha]$ is a witness for W in the instance of *Strong Collection*

$$\forall X (\forall U \in X \exists Y G[U,Y] \to \exists W (\forall U \in X \exists Y \in W G[U,Y] \wedge \forall Y \in W \exists U \in X G[U,Y])).$$

For realization of $\forall U \in \alpha \exists Y \in sc[\mathfrak{r}, \alpha] G[U,Y] \wedge \forall Y \in sc[\mathfrak{r}, \alpha] \exists U \in \alpha G[U,Y]$, it's convenient to use modified realisabilities $\underline{\mathrm{rn}}^\forall$ and $\underline{\mathrm{rn}}^\exists$ (Definition 2.12 and Lemma 2.6): then everything follows from the definition of sc.

□

Lemma 3.8 *Fullness*
Fullness axiom is realizable.

Proof. Assume $\mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta]$. Consider the following formula $Fn[\alpha, \beta, f]$:

$$\forall u \,\varepsilon\, \alpha(\mathsf{ln} u = 1 \to fu \,\varepsilon\, \beta \wedge \mathsf{ln}(fu) = 1). \tag{3.39}$$

By **ECA** there exists a name $fn[\alpha, \beta]$ s.t.

$$f \,\varepsilon\, fn[\alpha, \beta] \leftrightarrow Fn[\alpha, \beta, f]. \tag{3.40}$$

Now consider a formula $Fl[\alpha, \beta, c]$:

$$\begin{gathered}\mathsf{ln} c = 0 \ \bigvee\ \mathsf{ln} c = 1 \wedge \mathsf{ct} c \,\varepsilon\, fn[\alpha, \beta] \ \bigvee\ \mathsf{ln} c \neq 0 \wedge \mathsf{ln} c \neq 1 \wedge \mathsf{ct} c \,\varepsilon\, fn[\alpha, \beta] \wedge \\ \mathsf{head}(\mathsf{bodyt} c) \,\varepsilon\, \alpha \wedge \mathsf{fcut}(c, 2) \,\varepsilon\, \mathsf{opt}(\mathsf{str}(\alpha, \mathsf{head}(\mathsf{bodyt} c)), \mathsf{str}(\beta, \mathsf{ct} c(\mathsf{head}(\mathsf{bodyt} c)))).\end{gathered} \tag{3.41}$$

By **ECA** there exists a name $fl[\alpha, \beta]$ s.t.

$$c \,\varepsilon\, fl[\alpha, \beta] \leftrightarrow c \,\varepsilon\, \mathsf{seq} \wedge Fl[\alpha, \beta, c]. \tag{3.42}$$

This $fl[\alpha, \beta]$ is a witness for Z in the *Fullness* axiom

$$\forall X \forall Y \exists Z \mathrm{Full}[Z, X, Y].$$

Let's check that we have a realization of

$$\mathrm{Full}[\alpha, \beta, fl[\alpha, \beta]] \ \equiv\ \forall W \in fl[\alpha, \beta] \mathbf{mv}[W, A, B] \wedge \forall R(\mathbf{mv}[R, A, B] \to \exists S \in fl[\alpha, \beta] \forall W \in S (W \in R)).$$

By Definition 2.10

$$\mathfrak{r}\ \underline{\mathrm{rn}}^{\mathcal{E}} \mathbf{mv}[\rho, \alpha, \beta] \ \equiv\ \mathsf{p}_0\mathfrak{r}\ \underline{\mathrm{rn}}^{\mathcal{E}} \forall Q \in \rho \exists U \in \alpha \exists V \in \beta (Q = \langle U, V\rangle) \ \wedge\ \mathsf{p}_1\mathfrak{r}\ \underline{\mathrm{rn}}^{\mathcal{E}} \forall U \in \alpha \exists V \in \beta (\langle U, V\rangle \in \rho). \tag{3.43}$$

Given $w \,\varepsilon\, fl[\alpha, \beta]$, $\mathsf{ln} w = 1$, namely $w = \mathsf{sg} f$, $f \,\varepsilon\, fn[\alpha, \beta]$, we have an elementary realization of $\forall Q \in \mathsf{str}(fl[\alpha, \beta], w) \exists U \in \alpha \exists V \in \beta (Q = \langle U, V\rangle)$: if we take $u := q$ and $v := fq$, then we have to realize $\mathsf{str}(fl[\alpha, \beta], w * q) = \langle \mathsf{str}(\alpha, q), \mathsf{str}(\beta, fq)\rangle$, i.e. $\mathsf{opt}(\mathsf{str}(\alpha, q), \mathsf{str}(\beta, fq)) = \langle \mathsf{str}(\alpha, q), \mathsf{str}(\beta, fq)\rangle$, which follows from Proposition 3.1b), and of $\forall U \in \alpha \exists V \in \beta (\langle U, V\rangle \in \mathsf{str}(fl[\alpha, \beta], w))$: take $v := f(u)$. Also, if $\mathrm{Set}[\rho]$ and $\mathfrak{r}\underline{\mathrm{rn}}^{\mathcal{E}} \mathbf{mv}[\rho, \alpha, \beta]$, then $\mathsf{p}_1\mathfrak{r}$ provides us with a function $f \,\varepsilon\, fn[\alpha, \beta]$ and a realization of $\langle \mathsf{str}(\alpha, u), \mathsf{str}(\beta, fu)\rangle \in \rho$, and then we take $s := \mathsf{sg} f$, and by construction 3.41 and Proposition 3.1b) realize $\forall W \in \mathsf{str}(fl[\alpha, \beta], \mathsf{sg} f)(W \in \rho)$.

□

All the lemmas proved in this section, together with Theorem 1 in Section 2, give us

Theorem 2
a) **CZF** *is realizable in* $\mathbf{T_0}$*;*
b) **CZF**$\upharpoonright$ *is realizable in* $\mathbf{T_0}\upharpoonright$*;*
c) **CZF** − *Strong Collection is realizable in* **EET + IG***;*
d) **CZF**$\upharpoonright$ − *Strong Collection is realizable in* $(\mathbf{EET}+\mathbf{IG})\upharpoonright$.

4 Regularity, inaccessibility, mahloness

There is a perfect intuitive match between higher notions in Constructive Set Theory, starting with *regularity*, and higher universe properties in Explicit Mathematics. To explain this match formally, we need to be able to pass from a universe υ, having some higher property $\mathcal{F}$, to a *universal set* $\mathsf{us}\upsilon$, having corresponding higher set-property $\mathcal{F}'$. If υ names a universe, such a $\mathsf{us}\upsilon$ is constructed by collecting all sets α, $\alpha \,\varepsilon\, \upsilon$, indexed by α, into a single tree. In the definition below we are using the fact that in υ, making use of Join over υ, the formula Set$[x]$, Definition 2.3, can be replaced by an elementary formula.

Definition 4.1 us – universal set
Let υ *name a universe.*

1. *We define names*

$$j[\upsilon] := \mathsf{j}(\upsilon, \lambda x.x), \;\; j_{\mathsf{ig}}[\upsilon] := \mathsf{j}(\upsilon, \lambda x.\mathsf{ig}x). \tag{4.1}$$

2. *An elementary formula* $\mathrm{Set}^{-\mathcal{N}}[x;\upsilon]$ *is defined as*

$$\begin{gathered}\forall y(\langle x,y\rangle \,\varepsilon\, j[\upsilon] \to y \,\varepsilon\, \mathsf{seq}) \wedge \langle x, \mathsf{nil}\rangle \,\varepsilon\, j[\upsilon] \;\wedge \\ \forall y \forall z(\langle x,y\rangle \,\varepsilon\, j[\upsilon] \wedge y \sqsupset z \to \langle x,z\rangle \,\varepsilon\, j[\upsilon]) \wedge \forall y(\langle x,y\rangle \,\varepsilon\, j[\upsilon] \to \langle x,y\rangle \,\varepsilon\, j_{\mathsf{ig}}[\upsilon]).\end{gathered} \tag{4.2}$$

3. *By* **ECA** *we define a name*

$$u[\upsilon] := \{x \mid x \,\varepsilon\, \upsilon \wedge \mathrm{Set}^{-\mathcal{N}}[x;\upsilon]\}. \tag{4.3}$$

4. *Finally by* **ECA** *and* λ*-abstraction we set*

$$\mathsf{us}\upsilon := \{x \mid x \,\varepsilon\, \mathsf{seq} \wedge (\mathsf{ln}x = 0 \vee (\mathsf{ln}x \neq 0 \wedge \langle \mathsf{ct}(\mathsf{head}x), \mathsf{bodyt}x\rangle \,\varepsilon\, \mathsf{j}(u[\upsilon], \lambda x.x)))\}. \tag{4.4}$$

Lemma 4.1 $\mathrm{Set}^{-\mathcal{N}}[x;\upsilon] \leftrightarrow \mathrm{Set}[x]$
If υ *names a universe, then*

$$\mathrm{Set}^{-\mathcal{N}}[x;\upsilon] \leftrightarrow \mathrm{Set}[x] \wedge x \,\varepsilon\, \upsilon. \tag{4.5}$$

Proof. $\Rightarrow$. Assuming $\mathrm{Set}^{-\mathcal{N}}[x;\upsilon]$, $x \,\varepsilon\, \upsilon$ follows from $\langle x,\mathsf{nil}\rangle \,\varepsilon\, j[\upsilon]$. Now by Join $\mathrm{Set}^{-\mathcal{N}}[x;\upsilon]$ (4.2) and Set$[x]$ (2.20) say the same thing, except for $\mathcal{N}[x]$, which is required by Set$[x]$ and follows from $x \,\varepsilon\, \upsilon$.
$\Leftarrow$. $\mathrm{Set}^{-\mathcal{N}}[x;\upsilon]$ follows from $\mathrm{Set}[x] \wedge x \,\varepsilon\, \upsilon$ by Join.

□

From the Definition 4.1 and the previous Lemma we obtain the following:

Lemma 4.2 Universal set lemma
If υ *names a universe, then the following holds:*
a) $\mathrm{Set}[\mathsf{us}\upsilon]$*;*
b) $\alpha \,\varepsilon\, \upsilon \wedge \mathrm{Set}[\alpha] \to \langle \mathsf{sg}\alpha, \mathsf{ID}\rangle \;\underline{\mathrm{rn}}\; (\alpha \in \mathsf{us}\upsilon)$*;*
c) $x \,\varepsilon\, \mathsf{us}\upsilon \wedge \mathsf{ln}x = 1 \to \mathsf{ct}x \,\varepsilon\, \upsilon \wedge \mathrm{Set}[\mathsf{ct}x] \wedge \mathsf{str}(\mathsf{us}\upsilon, x) \doteq \mathsf{ct}x$.

Importance of the notion of universal set is that, if υ names a universe, then every name construction which we carried out in Lemmas 3.3–3.8 is reflected by this set (= on first nodes of this tree). More exactly, we have the following lemma.

Lemma 4.3

$$\mathcal{U}[v] \wedge \alpha, \beta \,\varepsilon\, v \wedge \mathrm{Set}[\alpha] \wedge \mathrm{Set}[\beta] \rightarrow t[\alpha,\beta] \,\varepsilon\, v \wedge \mathrm{Set}[t[\alpha,\beta]] \wedge \langle \mathsf{sg}t[\alpha,\beta], \mathsf{ID}\rangle \,\underline{\mathrm{rn}}\, (t[\alpha,\beta] \in \mathsf{us}v) \tag{4.6}$$

for $t[\alpha,\beta] ::= \mathsf{pt}\alpha\beta, u[\alpha], \inf, s[\alpha], sc[\mathfrak{r},\alpha], \mathit{fl}[\alpha,\beta]$ *as defined in the proofs of Lemmas 3.3–3.8.*

Proof. $\mathrm{Set}[t[\alpha,\beta]]$ is established in Lemmas 3.3–3.8. $(t[\alpha,\beta]\varepsilon v)$-part follows from the fact that t is built up by Elementary Comprehension and Join, and universes are closed under j and name generators (Definition 1.2). $\underline{\mathrm{rn}}$ -part now follows from universal set Lemma 4.2b). □

The first set-theoretic property, where the notion of universal set will come into use, is that of *regularity.*

Definition 4.2 Regular set
$\mathrm{Reg}[A]$ *is the following formula of* $\mathcal{L}_\in$*:*

$$\mathrm{Tran}[A] \wedge \forall C \in A \forall R \subseteq C \times A$$
$$\big(\forall X \in C \exists Y \in A(\langle X,Y\rangle \in R) \rightarrow \exists B \in A(\forall X \in C \exists Y \in B(\langle X,Y\rangle \in R) \wedge \forall Y \in B \exists X \in C(\langle X,Y\rangle \in R))\big),$$

where $\mathrm{Tran}[A]$ *stands for* $\forall X \in A \forall Y \in X (Y \in A)$.

In fact, the formula $\mathrm{Reg}[A]$ says that A is transitive and closed under an instance of *Strong Collection* for a specific formula $G[X,Y]$, namely $G[X,Y] \equiv \langle X,Y\rangle \in R$, for $R \subseteq C \times A$, $C \in A$.

Definition 4.3 REA
REA *is an axiom*

$$\forall X \exists Y (X \subseteq Y \wedge \mathrm{Reg}[Y]).$$

Lemma 4.4 $\mathrm{Reg}[\mathsf{us}v]$
If v *names a universe then* $\mathrm{Reg}[\mathsf{us}v]$ *is realizable in* $\mathbf{T_0}\!\upharpoonright$.

Proof.
We have to realize two formulas:

$$\forall X \in \mathsf{us}v \forall Y \in X (Y \in \mathsf{us}v) \tag{4.7}$$

and

$$\begin{aligned}\forall C \in \mathsf{us}v \forall R \subseteq C \times \mathsf{us}v \big(\forall X \in C \exists Y \in \mathsf{us}v(\langle X,Y\rangle \in R) \rightarrow \\ \exists B \in \mathsf{us}v(\forall X \in C \exists Y \in B(\langle X,Y\rangle \in R) \wedge \forall Y \in B \exists X \in C(\langle X,Y\rangle \in R))\big).\end{aligned} \tag{4.8}$$

4.7: Assume $x \,\varepsilon\, \mathsf{us}v \wedge \mathsf{ln}x = 1 \wedge y \,\varepsilon\, \mathsf{str}(\mathsf{us}v, x) \wedge \mathsf{ln}y = 1$. By Lemma 4.2c) we have

$$\mathsf{ct}x \,\varepsilon\, v \wedge \mathrm{Set}[\mathsf{ct}x] \wedge y \,\varepsilon\, \mathsf{ct}x \tag{4.9}$$

and

$$\mathsf{str}(\mathsf{str}(\mathsf{us}v, x), y) \doteq \mathsf{str}(\mathsf{ct}x, y). \tag{4.10}$$

$\mathsf{str}(\mathsf{ct}x, y) \,\varepsilon\, v$, since $\mathsf{ct}x \,\varepsilon\, v$ and v is closed under str. Also, $\mathrm{Set}[\mathsf{str}(\mathsf{ct}x, y)]$, since $y \,\varepsilon\, \mathsf{ct}x$. By Lemma 4.2b)

$$\langle \mathsf{sg}(\mathsf{str}(\mathsf{ct}x, y)), \mathsf{ID}\rangle \,\underline{\mathrm{rn}}\, (\mathsf{str}(\mathsf{ct}x, y) \in \mathsf{us}v), \tag{4.11}$$

which together with 4.10 yields

$$\langle \mathsf{sg}(\mathsf{str}(\mathsf{ct}x, y)), \mathsf{ID}\rangle \,\underline{\mathrm{rn}}\, (\mathsf{str}(\mathsf{str}(\mathsf{us}v, x), y) \in \mathsf{us}v). \tag{4.12}$$

4.8: Assume $c \,\varepsilon\, \mathsf{us}v$, $\mathsf{ln}c = 1$, $\mathrm{Set}[\rho] \wedge \mathfrak{p} \,\underline{\mathrm{rn}}\, (\rho \subseteq \mathsf{str}(\mathsf{us}v, c) \times \mathsf{us}v)$ and

$$\mathfrak{r} \,\underline{\mathrm{rn}}^{\mathcal{E}} \forall X \in \mathsf{str}(\mathsf{us}v, c) \exists Y \in \mathsf{us}v(\langle X,Y\rangle \in \rho), \tag{4.13}$$

which reads

$$\begin{aligned}&\forall x \,\varepsilon\, \mathsf{str}(\mathsf{us}v, c)(\mathsf{ln}x = 1 \rightarrow \mathfrak{r}x\!\downarrow \underline{\mathrm{rn}}^{\mathcal{E}} \exists Y \in \mathsf{us}v(\langle \mathsf{str}(\mathsf{str}(\mathsf{us}v, c), x), Y\rangle \in \rho)) \equiv \\ &\forall x \,\varepsilon\, \mathsf{str}(\mathsf{us}v, c)(\mathsf{ln}x = 1 \rightarrow \\ &\quad \mathsf{p}_0(\mathfrak{r}x) \,\varepsilon\, \mathsf{us}v \wedge \mathsf{ln}(\mathsf{p}_0(\mathfrak{r}x)) = 1 \wedge \mathsf{p}_1(\mathfrak{r}x) \,\underline{\mathrm{rn}}\, (\langle \mathsf{str}(\mathsf{str}(\mathsf{us}v, c), x), \mathsf{str}(\mathsf{us}v, \mathsf{p}_0(\mathfrak{r}x))\rangle \in \rho)).\end{aligned} \tag{4.14}$$

By Lemma 4.2c)

$$\mathsf{ct}c \,\varepsilon\, \upsilon \wedge \mathrm{Set}[\mathsf{ct}c] \wedge \mathsf{str}(\mathsf{us}\upsilon, c) \doteq \mathsf{ct}c, \tag{4.15}$$

i.e. we have

$$\begin{aligned} &\forall x \,\varepsilon\, \mathsf{ct}c(\mathsf{ln}x = 1 \;\rightarrow \\ &\qquad \mathsf{p_0}(\mathfrak{r}x) \,\varepsilon\, \mathsf{us}\upsilon \wedge \mathsf{ln}(\mathsf{p_0}(\mathfrak{r}x)) = 1 \wedge \mathsf{p_1}(\mathfrak{r}x) \;\underline{\mathrm{rn}}\; (\langle \mathsf{str}(\mathsf{str}(\mathsf{us}\upsilon, c), x), \mathsf{str}(\mathsf{us}\upsilon, \mathsf{p_0}(\mathfrak{r}x))\rangle \in \rho)). \end{aligned} \tag{4.16}$$

Taking a name $t[c] :\doteq \{x \mid x \,\varepsilon\, \mathsf{ct}c \wedge \mathsf{ln}x = 1\}$ and an operation $f := \lambda x.\mathsf{ct}(\mathsf{p_0}(\mathfrak{r}x))$, we have $t \,\varepsilon\, \upsilon \wedge f : t \mapsto \upsilon$, so that $\mathsf{j}(t, f) \,\varepsilon\, \upsilon \wedge \forall x \,\varepsilon\, t\mathrm{Set}[fx]$. Now the proof is completed in the same way as of Lemma 3.7: we take a name $sc'[\mathfrak{r}, c]$ defined by **ECA** to satisfy

$$y \,\varepsilon\, sc'[\mathfrak{r}, c] \leftrightarrow y \,\varepsilon\, \mathsf{seq} \wedge (y = \mathsf{nil} \vee \langle \mathsf{head}y, \mathsf{bodyt}y\rangle \,\varepsilon\, \mathsf{j}(t, f)), \tag{4.17}$$

as a witness for B. Note in addition that we have a realization of $sc' \in \mathsf{us}\upsilon$, since $sc' \,\varepsilon\, \upsilon$, $\mathrm{Set}[sc']$, and therefore by Lemma 4.2b) the formula $sc' \in \mathsf{us}\upsilon$ is realizable.

□

Theorem 3
a) $\mathbf{CZF} + \mathbf{REA}$ *is realizable in* $\mathbf{T_u}$*;*
b) $\mathbf{CZF}{\restriction} + \mathbf{REA}$ *is realizable in* $\mathbf{T_u}{\restriction}$*.*

Proof.
a) **CZF** is realizable in $\mathbf{T_u}$ since by Theorem 2a) it's realizable in $\mathbf{T_0}$ and $\mathbf{T_0}$ is a subsystem of $\mathbf{T_u}$. So we have to concentrate on **REA**.
If α is a name for X, $\mathrm{Set}[\alpha]$, we take $\mathsf{us}(\mathsf{u}\alpha)$ as a name for Y.
Since $\mathcal{U}[\mathsf{u}\alpha]$ by VII, by the previous Lemma we have a realization of $\mathrm{Reg}[\mathsf{us}(\mathsf{u}\alpha)]$, and it remains only to (elementarily) realize $\alpha \subseteq \mathsf{us}(\mathsf{u}\alpha) \;\equiv\; \forall X \in \alpha(X \in \mathsf{us}(\mathsf{u}\alpha))$.
$\mathsf{str}(\alpha, x) \,\varepsilon\, \mathsf{u}\alpha$, since $\alpha \,\varepsilon\, \mathsf{u}\alpha$ by VII and str is built by **ECA**. Since $x \,\varepsilon\, \alpha$, by Lemma 2.1 $\mathrm{Set}[\mathsf{str}(\alpha, x)]$. Then by Lemma 4.2b) $\langle \mathsf{sg}(\mathsf{str}(\alpha, x)), \mathsf{ID}\rangle \;\underline{\mathrm{rn}}\; (\mathsf{str}(\alpha, x) \in \mathsf{us}(\mathsf{u}\alpha))$.

b) As a), using Theorem 2b) instead of 2a).

□

Corollary.
a) $|\mathbf{CZF} + \mathbf{REA}| \leq |\mathbf{T_u}| \leq |\mathbf{KPi}|$;
b) $|\mathbf{CZF}{\restriction} + \mathbf{REA}| \leq |\mathbf{T_u}{\restriction}| \leq |\mathbf{KPi}{\restriction}|$.

Proof. This follows from the previous theorem and parts 3 and 2 of [JStu, Theorem 6] and [Jnm, Theorem 11].

□

For realizing Mahlo axioms in Constructive Set Theory, we need an operation set, building a set out of an arbitrary name "by brute force".

Definition 4.4 Set-forming operation set
By **ECA** *we define a name* $t[\alpha]$ *so that*

$$t[\alpha] \doteq \{x \mid \exists z \,\varepsilon\, \alpha(z \sqsupseteq x)\}. \tag{4.18}$$

Then by **ECA** *and* λ*-abstraction we define*

$$\mathsf{set}\alpha :\doteq \{x \mid x = \mathsf{nil} \vee x \,\varepsilon\, \mathsf{ig}t[\alpha]\}. \tag{4.19}$$

We have an obvious

Lemma 4.5 *For each name* α *we have*

$$\mathrm{Set}[\mathsf{set}\alpha] \;\wedge\; (\mathrm{Set}[\alpha] \rightarrow \mathsf{set}\alpha \doteq \alpha).$$

Definition 4.5 Inaccessible set, $\mathrm{In}[A]$
$\mathrm{In}[A]$ *is a formula*

$$\mathrm{Reg}[A] \wedge \mathbf{REA}^A \wedge \mathit{Pair}^A \wedge \mathit{Union}^A \wedge \mathit{Infinity}^A \wedge \wedge_{i=\overline{1,N}}(\mathit{Bounded\ Separation})_i^A \wedge \mathit{Fullness}^A, \tag{4.20}$$

where $(\mathit{Bounded\ Separation})_i$ *is the* i*-th formula in the finite formalisation of Bounded Separation (see Section 3).*

Definition 4.6 Mahlo schema, **CZFM**, **CZFM**$\upharpoonright$
M *is the following schema:*

$$\forall X \exists Y F[X,Y] \rightarrow \exists I(\mathrm{In}[I] \wedge \forall X \in I \exists Y \in I F[X,Y])$$
for all formulas $F[X,Y]$.

By **CZFM** *we denote a theory* **CZF** + **REA** + **M**, **CZFM**$\upharpoonright$:= **CZF**$\upharpoonright$ + **REA** + **M**.

Definition 4.7 Inaccessible universe
A universe ι *is* $\{\mathsf{u}', \mathsf{i}'\}$-inaccessible *iff*

$$\mathsf{u}' : \iota \mapsto \iota \wedge \mathsf{i}' : \iota^2 \mapsto \iota \wedge \forall \alpha \,\varepsilon\, \iota(\mathsf{u}'\alpha \doteq \mathsf{u}\alpha) \wedge \forall \alpha \,\varepsilon\, \iota \forall \beta \,\varepsilon\, \iota(\mathsf{i}'\alpha\beta \doteq \mathsf{i}\alpha\beta). \tag{4.21}$$

Lemma 4.6 In[usι]
If a universe ι *is* $\{\mathsf{u}', \mathsf{i}'\}$*-inaccessible then the formula* In[usι] *is realizable in* $\mathbf{T_u}\upharpoonright$.

Proof. Assume a universe ι to be $\{\mathsf{u}', \mathsf{i}'\}$-inaccessible. Since ι names a universe, all conjuncts in the formula In[usι] (see 4.20), except Reg[usι] and $\mathbf{REA}^{\mathsf{us}\iota}$, are realizable by Lemma 4.3. Additionally, Reg[usι] is realizable by Lemma 4.4.
It remains to elementarily realize

$$\mathbf{REA}^{\mathsf{us}\iota} \equiv \forall X \in \mathsf{us}\iota \exists Y \in \mathsf{us}\iota(X \subseteq Y \wedge \mathrm{Reg}[Y]). \tag{4.22}$$

We start with $x \,\varepsilon\, \mathsf{us}\iota$, $\mathsf{In}x = 1$. As before, by Lemma 4.2c), $\mathsf{ct}x \,\varepsilon\, \iota \wedge \mathrm{Set}[\mathsf{ct}x]$, and, consequently, $\mathsf{u}'(\mathsf{ct}x) \,\varepsilon\, \iota \wedge \mathsf{i}'(\mathsf{ct}x, \sqsupset) \,\varepsilon\, \iota$. By 4.21

$$\mathsf{u}'(\mathsf{ct}x) \doteq \mathsf{u}(\mathsf{ct}x) \wedge \mathsf{i}'(\mathsf{ct}x, \sqsupset) \doteq \mathsf{ig}(\mathsf{ct}x), \tag{4.23}$$

so we also have $\mathcal{U}[\mathsf{u}'(\mathsf{ct}x)]$.
We define an operation us' in the same way as operation us in Definition 4.1, with the only difference that i is everywhere replaced by i'. Since $\mathsf{us}'(\mathsf{u}'(\mathsf{ct}x))$ is built from $\mathsf{u}'(\mathsf{ct}x)$ by Elementary Comprehension and Join, using closeness of ι under i', we have $\mathsf{us}'(\mathsf{u}'(\mathsf{ct}x)) \,\varepsilon\, \iota$ and

$$\mathsf{us}'(\mathsf{u}'(\mathsf{ct}x)) \doteq \mathsf{us}(\mathsf{u}'(\mathsf{ct}x)) \doteq \mathsf{us}(\mathsf{u}(\mathsf{ct}x)). \tag{4.24}$$

Elementary witness for Y is taken to be $\mathsf{sg}(\mathsf{us}'(\mathsf{u}'(\mathsf{ct}x)))$. Rewriting 4.22 with this value of Y, we need to find a realization of

$$\mathsf{str}(\mathsf{us}\iota, x) \subseteq \mathsf{str}(\mathsf{us}\iota, \mathsf{sg}(\mathsf{us}'(\mathsf{u}'(\mathsf{ct}x)))) \;\wedge\; \mathrm{Reg}[\mathsf{str}(\mathsf{us}\iota, \mathsf{sg}(\mathsf{us}'(\mathsf{u}'(\mathsf{ct}x))))], \tag{4.25}$$

which by 4.24 and Lemmas 4.2c) and 2.4 reduces in turn to realizability of

$$\mathsf{str}(\mathsf{us}\iota, x) \subseteq \mathsf{us}(\mathsf{u}(\mathsf{ct}x)) \wedge \mathrm{Reg}[\mathsf{us}(\mathsf{u}(\mathsf{ct}x))], \tag{4.26}$$

and

$$\mathsf{ct}x \subseteq \mathsf{us}(\mathsf{u}(\mathsf{ct}x)) \wedge \mathrm{Reg}[\mathsf{us}(\mathsf{u}(\mathsf{ct}x))]. \tag{4.27}$$

Realizability of the first conjunct of the latter formula follows as in the proof of Theorem 3, and of the second conjunct follows from Lemma 4.4.

□

Lemma 4.7 *Mahlo schema is realizable in* $\mathbf{T_m}\upharpoonright$.

Proof. Assume

$$\mathfrak{r}\, \underline{\mathrm{rn}}\, \forall X \exists Y G[X,Y]. \tag{4.28}$$

Consider a function $f := \lambda\alpha.\mathsf{p}_0(\mathfrak{r}(\mathsf{set}\alpha))$. Take $\iota := \mathsf{m}(\mathsf{nat}, \lambda\alpha.\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha)))$ and $\mathsf{u}' := \lambda\alpha.\mathsf{pt}_0(\mathsf{pt}_1(\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha))))$, $\mathsf{i}' := \lambda\alpha.\mathsf{pt}_1(\mathsf{pt}_1(\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha))))$. By VIII and Proposition 3.1c) ι is $\{\mathsf{u}', \mathsf{i}'\}$-inaccessible.
We take $\mathsf{us}\iota$ as a name for I. In[usι] is realizable by Lemma 4.6.
It remains to find a $\forall\exists$-realizer of

$$\forall X \in \mathsf{us}\iota \exists Y \in \mathsf{us}\iota G[X,Y].$$

If we take $f' := \lambda\alpha.\mathsf{pt}_0(\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha)))$, then

$$f' : \iota \mapsto \iota \wedge \forall\alpha(f'\alpha \doteq f\alpha). \tag{4.29}$$

Assuming $x \,\varepsilon\, \mathsf{us}\iota$, $\mathsf{ln}x = 1$, we have $\mathsf{ct}x \,\varepsilon\, \iota \wedge \mathrm{Set}[\mathsf{ct}x]$, $f'(\mathsf{ct}x) \,\varepsilon\, \iota$, and $\mathrm{Set}[f'(\mathsf{ct}x)]$, since $\mathrm{Set}[f(\mathsf{ct}x)]$. $\mathsf{sg}(f'(\mathsf{ct}x))$ is taken as an $\exists$-witness for Y. By Lemma 4.2,

$$\mathsf{sg}(f'(\mathsf{ct}x)) \,\varepsilon\, \mathsf{us}\iota,\ \mathsf{ln}(\mathsf{sg}(f'(\mathsf{ct}x))) = 1 \tag{4.30}$$

and

$$\begin{array}{c} \mathsf{str}(\mathsf{us}\iota, x) \doteq \mathsf{ct}x \,\varepsilon\, \iota, \\ \mathsf{str}(\mathsf{us}\iota, \mathsf{sg}(f'(\mathsf{ct}x))) \doteq f'(\mathsf{ct}x) \,\varepsilon\, \iota. \end{array} \tag{4.31}$$

From 4.28 we have

$$\mathsf{p}_1(\mathfrak{r}(\mathsf{set}(\mathsf{ct}x)))\downarrow \underline{\mathrm{rn}}\, G[\mathsf{set}(\mathsf{ct}x), \mathsf{p}_0(\mathfrak{r}(\mathsf{set}(\mathsf{ct}x)))]. \tag{4.32}$$

From the definitions of f' and f we have

$$f'(\mathsf{ct}x) \doteq f(\mathsf{ct}x) = \mathsf{set}(\mathsf{p}_0(\mathfrak{r}(\mathsf{set}(\mathsf{ct}x)))). \tag{4.33}$$

Since $\mathrm{Set}[\mathsf{p}_0(\mathfrak{r}(\mathsf{set}(\mathsf{ct}x)))]$ by Definition 2.6 and $\mathrm{Set}[\mathsf{ct}x]$, by Lemma 4.5 we have

$$\begin{array}{c} \mathsf{set}(\mathsf{ct}x) \doteq \mathsf{ct}x, \\ \mathsf{set}(\mathsf{p}_0(\mathfrak{r}(\mathsf{set}(\mathsf{ct}x)))) \doteq \mathsf{p}_0(\mathfrak{r}(\mathsf{set}(\mathsf{ct}x))). \end{array} \tag{4.34}$$

Now, combining 4.31–4.34 with Lemma 2.4, we obtain a realizer of $G[\mathsf{str}(\mathsf{us}\iota, x), \mathsf{str}(\mathsf{us}\iota, \mathsf{sg}(f'(\mathsf{ct}x)))]$.

□

Theorem 4
a) **CZFM** *is realizable in* $\mathbf{T_m}$*;*
b) **CZFM**$\restriction$ *is realizable in* $\mathbf{T_m}\restriction$*.*

Proof. This follows from Theorem 3 and the previous lemma. □

Corollary.
a) $|\mathbf{CZFM}| \leq |\mathbf{T_m}| \leq |\mathbf{KPM}|$;
b) $|\mathbf{CZFM}\restriction| \leq |\mathbf{T_m}\restriction| \leq |\mathbf{KPM}\restriction|$.

Proof. This follows from the previous theorem and parts 3 and 2 of [JStu, Theorem 10] and [Jnm, Theorem 13]. □

A little stronger form of mahloness in Constructive Set Theory, considered in [RaCZFM], is existence of a *Mahlo set.* As before, *Mahlo universe* M takes care of such a set.

Definition 4.8 Mahlo set, $\mathrm{M}[A]$
$\mathrm{M}[A]$ *is defined as*

$$\mathrm{In}[A] \wedge \forall R(\mathbf{mv}[R, A, A] \rightarrow \exists I \in A(\mathrm{In}[I] \wedge \forall X \in I \exists Y \in I(\langle X, Y\rangle \in I))). \tag{4.35}$$

Definition 4.9 Mahlo axiom, $\mathbf{CZFM}^+$, $\mathbf{CZFM}^+\restriction$
$\mathbf{M}^+$ *is an axiom*

$$\exists Z \mathrm{M}[Z].$$

$\mathbf{CZFM}^+$ *is a theory* $\mathbf{CZF} + \mathbf{REA} + \mathbf{M}^+$*,* $\mathbf{CZFM}^+\restriction := \mathbf{CZF}\restriction + \mathbf{REA} + \mathbf{M}^+$*.*

Realizability of the Mahlo axiom repeats realizability of the Mahlo schema in Lemma 4.7, just with a little extra considerations.

Lemma 4.8 *Mahlo axiom is realizable in* $\mathbf{T_M}\restriction$*.*

Proof. First of all, by IX,

$$\mathsf{u}:\mathsf{M} \mapsto \mathsf{M} \wedge \mathsf{i}:\mathsf{M}^2 \mapsto \mathsf{M}. \tag{4.36}$$

Therefore M is $\{\mathsf{u},\mathsf{i}\}$-inaccessible and by Lemma 4.6 we have a realization of In[usM]. usM is taken as a witness for Z.
Now assume Set[ρ] $\wedge\ \mathfrak{r}\ \underline{\mathrm{rn}}^{\mathcal{E}}\mathbf{mv}[\rho, \mathsf{usM}, \mathsf{usM}]$. Take $f := \lambda\alpha.\mathsf{ct}(\mathsf{p_1}\mathfrak{r}(\mathsf{sg}(\mathsf{set}\alpha)))$, then

$$f:\mathsf{M} \mapsto \mathsf{M}. \tag{4.37}$$

Take $\iota := \mathsf{m}(\mathsf{nat}, \lambda\alpha.\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha)))$ and $\mathsf{u}' := \lambda\alpha.\mathsf{pt_0}(\mathsf{pt_1}(\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha))))$, $\mathsf{i}' := \lambda\alpha.\mathsf{pt_1}(\mathsf{pt_1}(\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha))))$. By IX and Proposition 3.1c) we obtain that $\iota \,\varepsilon\, \mathsf{M}$ and ι is $\{\mathsf{u}', \mathsf{i}'\}$-inaccessible.
We now take $\mathsf{us}\iota$ as a witness for I. In[$\mathsf{us}\iota$] is realizable by Lemma 4.6. Since $\iota \,\varepsilon\, \mathsf{M}$, we also have $\mathsf{us}\iota \,\varepsilon\, \mathsf{M}$. Since Set[$\mathsf{us}\iota$] by Lemma 4.2a), we have a realization of $\mathsf{us}\iota \in \mathsf{usM}$.
Finally, we need to find an elementary realizer of

$$\forall X \in \mathsf{us}\iota \exists Y \in \mathsf{us}\iota(\langle X, Y\rangle \in \mathsf{us}\iota).$$

If we take $f' := \lambda\alpha.\mathsf{pt_0}(\mathsf{opt}(f\alpha, \mathsf{opt}(\mathsf{u}\alpha, \mathsf{i}\alpha)))$, then

$$f':\iota \mapsto \iota. \tag{4.38}$$

We define an operation set' in the same way as the operation set in Definition 4.4, with the only difference that i is everywhere replaced by i'. Now assuming $x \,\varepsilon\, \mathsf{us}\iota$, $\mathsf{ln}x = 1$, by Lemmas 4.2 and 4.5 we have

$$\mathsf{sg}(\mathsf{set}'(f'(\mathsf{ct}x))) \,\varepsilon\, \mathsf{us}\iota,\ \ \mathsf{ln}(\mathsf{sg}(\mathsf{set}'(f'(\mathsf{ct}x)))) = 1 \tag{4.39}$$

and

$$\mathsf{opt}(\mathsf{str}(\mathsf{us}\iota, x), \mathsf{str}(\mathsf{us}\iota, \mathsf{sg}(\mathsf{set}'(f'(\mathsf{ct}x))))) \doteq \mathsf{opt}(\mathsf{ct}x, \mathsf{set}'(f'(\mathsf{ct}x))) \,\varepsilon\, \iota, \tag{4.40}$$

which by Proposition 3.1b) and Lemma 2.4 is sufficient for realizing

$$\langle \mathsf{str}(\mathsf{us}\iota, x), \mathsf{str}(\mathsf{us}\iota, \mathsf{sg}(\mathsf{set}'(f'(\mathsf{ct}x))))\rangle \in \mathsf{us}\iota.$$

□

Theorem 5
a) $\mathbf{CZFM}^+$ *is realizable in* $\mathbf{T_M}$;
b) $\mathbf{CZFM}^+\!\upharpoonright$ *is realizable in* $\mathbf{T_M}\!\upharpoonright$.

Proof. This follows from Theorem 3 and Lemma 4.8. □

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